

微 積 分

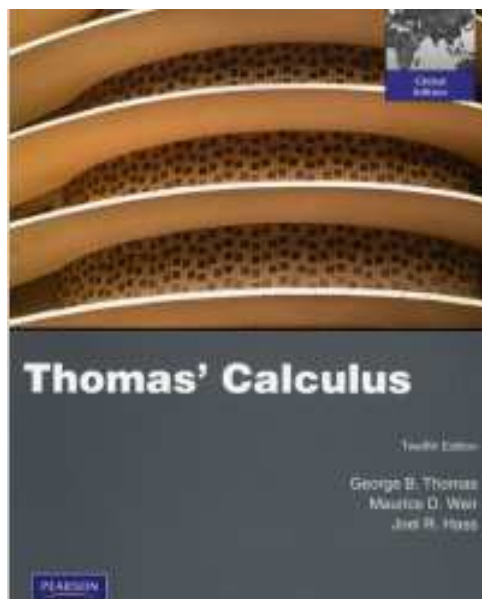
THOMAS' CALCULUS 12/E

吳漢銘 國立臺北大學統計學系

開課班級: 通訊 1/電機 1/智財學程

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附錄：105 學年第 1 學期小考題、期中考題、期末考題

叮嚀

- A. 教不完是正常; 考不好是日常; 平常就要唸書!
- B. 上課以「互相尊重」為最高原則並盡到「告知老師」的義務。
- C. 上課可小聲討論、可上廁所安靜去回、可飲食。(但請一定要維護教室整潔)
- D. 四不一要: 「上課不聊天, 睡覺不趴著, 手機不要滑, 考試不作弊, 要認真。」

THOMAS' CALCULUS (12/E)

2.1 Rates of Change and Tangents to Curves

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授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

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1 Average and Instantaneous Speed


1.1 **Average speed** during an interval of time: divide the _____ covered by the _____ elapsed.

$$\frac{\Delta y}{\Delta t} = \underline{\hspace{2cm}}$$

1.2 **Average speed** over a time interval $[t_0, t_0 + h]$ having length _____:

$$\frac{\Delta y}{\Delta t} = \underline{\hspace{2cm}}$$

1.3 The unit of measure is length per unit time; e.g., km/hr, m/sec.

 **Ex. 1** (example1, p40)

A rock breaks loose from the top of a tall cliff. What is its average speed (a) during the first 2 sec of fall? (b) during the 1-sec interval between second 1 and second 2.

sol:

 Ex. 2 (example2, p40)

Find the speed of the falling rock in Ex. 1 at $t = 1$ and $t = 2$ sec.

sol:

2 Average Rates of Change and Secant Lines

2.1 *Definition: Average Rate of Change over an Interval*

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0$$

2.2 The rate of change of f over $[x_1, x_2]$ is the _____ through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$.

2.3 The slope of a **secant** to the graph $y = f(x)$ is _____ (i.e., the average rate of change of f over the interval $[x_1, x_2]$). (如下圖)

 **Ex. 3** (example3, p42)

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

sol:

3 Instantaneous Rates of Change and Tangent Lines

3.1 **Instantaneous speed** at t_0 = average speed during _____ as _____.

3.2 The _____ corresponds to the slope of a _____;

3.3 The _____ corresponds to the slope of the _____.

3.4 The limiting values of average rates is the _____ (the rate at $t = t_0$).

實習課練習 (EXERCISE 2.1)

4. Find the average rate of change of the function g over the given intervals.
 $g(x) = 2 + \cos t$; (a) $[0, \pi]$ (b) $[-\pi, \pi]$
5. Find the average rate of change of the function $R(\theta) = \sqrt{4\theta + 1}$ over the given intervals $[0, 2]$.
12. Find the slope of the curve $y = 2 - x^3$ at the given point $P(1, 1)$, and an equation of the tangent line at $P(1, 1)$.
19. Let $g(x) = \sqrt{x}$ for $x \geq 0$. (a) Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$, and $[1, 1 + h]$. (b) Calculate the limit as h approaches zero.

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2.2 Limit of a Function and Limit Laws

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授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>**1 Limits of Function Values**

1.1 Let $f(x)$ be defined on an _____, except possibly at x_0 itself. If $f(x)$ get close to L for all x sufficiently close to x_0 , we say that f approaches the _____ L as x approaches x_0 , and we write

_____ ,
 read as _____ .

 **Ex. 1** (example1, p46)

How does the function $f(x) = \frac{x^2 - 1}{x - 1}$ behave near $x = 1$?

sol:

 **Ex. 2** (example3, p48)

(a) If f is the **identity function** _____, then for any value of x_0

$$\lim_{x \rightarrow x_0} f(x) = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

(b) If f is the **constant function** _____, then for any value of x_0

$$\lim_{x \rightarrow x_0} f(x) = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

 **Ex. 3** (example4, p48)

A function may fail to have a limit at a point in its domain.

Discuss the behavior of the following functions as $x \rightarrow 0$.

$$(a) U(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

$$(b) g(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$$(c) f(x) = \begin{cases} 0, & x \leq 0, \\ \sin \frac{1}{x}, & x > 0. \end{cases}$$

sol:

2 The Limit Laws

2.1 Theorem 1: Limit Laws

If L, M, c and k are _____ and _____ and _____, then _____

(a) sum rule: _____ .

(b) difference rule: _____ .

(c) constant multiple rule: _____ .

(d) product rule: _____ .

(e) quotient rule: _____ .

(f) power rule: if r and s are integers with no common factor and $s \neq 0$, then _____, provided that $L^{r/s}$ is a real number.

(g) root rule: _____, $n \in \mathbb{N}^+$.

2.2 Theorem 2: Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$

2.3 Theorem 3: Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \underline{\hspace{2cm}}.$$

 **Ex. 4** (example5, p50)

Find the following limits:

(a) $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3)$, (b) $\lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5}$, (c) $\lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$.

sol:

3 Eliminating Zero Denominators Algebraically


3.1 Canceling a _____.

3.2 Creating and canceling a _____.

 **Ex. 5** (example7, p51)

Evaluate $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$.

sol:

 **Ex. 6** (example9, p52)

Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.


sol:

4 The Sandwich Theorem

4.1 *Theorem 4: The Sandwich Theorem (The Squeeze Theorem, The Pinching Theorem)*


Suppose that _____ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that _____, then _____.

4.2 (a) $\lim_{\theta \rightarrow 0} \sin \theta = \underline{\hspace{2cm}}$. (b) $\lim_{\theta \rightarrow 0} \cos \theta = \underline{\hspace{2cm}}$.

 **Ex. 7** (example10, p53)

Given that $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$, for all $x \neq 0$, find $\lim_{x \rightarrow 0} u(x)$.

sol:

 **Ex. 8** (example11, p53)

For any function f , $\lim_{x \rightarrow c} |f(x)| = 0$ implies $\lim_{x \rightarrow c} f(x) = 0$.

sol:

實習課練習 (EXERCISE 2.2)

17. Find the limit: $\lim_{h \rightarrow -1} 3(2x - 1)^2$.

22. Find the limit: $\lim_{h \rightarrow 0} \frac{\sqrt{5h+4} - 2}{h}$.

36. Find the limit: $\lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}}$.

41. Find the limit: $\lim_{x \rightarrow -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3}$.

47. Find the limit: $\lim_{x \rightarrow 0} \frac{1 + x + \sin x}{3 \cos x}$.

63. If $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$ for $-1 \leq x \leq 1$, find $\lim_{x \rightarrow 0} f(x)$.

78. If $\lim_{x \rightarrow -2} \frac{f(x)}{x^2} = 1$, find (a) $\lim_{x \rightarrow -2} f(x)$, (b) $\lim_{x \rightarrow -2} \frac{f(x)}{x}$.

79. (a) If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 3$, find $\lim_{x \rightarrow 2} f(x)$.

(b) If $\lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 4$, find $\lim_{x \rightarrow 2} f(x)$.

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2.3 The Precise Definition of a Limit

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授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>**1 Definition of Limit**1.1 *Definition: Limit of a Function*

Let $f(x)$ be defined on an open interval about x_0 , except possibly at x_0 itself. We say that _____, and we write _____

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x ,

$$\text{_____} \Rightarrow \text{_____}$$

1.2 說明圖示如下:

 **Ex. 1** (example2, p59)

Show that $\lim_{x \rightarrow 1} (5x - 3) = 2$.

sol:

 **Ex. 2** (example3, p60)

Prove (a) $\lim_{x \rightarrow x_0} x = x_0$, (b) $\lim_{x \rightarrow x_0} k = k$.

sol:


2 Find δ Algebraically for a Given f , L , x_0 and $\epsilon > 0$

Finding a _____

2.1 Solve the inequality _____ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.


2.2 Find a value of _____ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b)

2.3 The inequality _____ will hold for all $x \neq x_0$ in this δ -interval.

 Ex. 3 (example4, p60)

Show that $\lim_{x \rightarrow 5} \sqrt{x-1} = 2$.

sol:

 Ex. 4 (example5, p61)

Prove that $\lim_{x \rightarrow 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2. \\ 1, & x = 2. \end{cases}$

sol:

實習課練習 (EXERCISE 2.3)

40. Prove the limit: $\lim_{x \rightarrow 0} \sqrt{4-x} = 2$.

43. Prove the limit: $\lim_{x \rightarrow 1} \frac{1}{x} = 1$.

48. Prove that $\lim_{x \rightarrow 0} f(x) = 0$ if $f(x) = \begin{cases} 2x, & x < 0. \\ x/2, & x \geq 0. \end{cases}$

49. Prove the limit: $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$.

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2.4 One-Sided Limits

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1 One-Sided Limits

1.1 Two-sided limits: _____ from _____.

1.2 Right-hand limit: _____.

("_____": consider only values of x greater than c .)

1.3 Left-hand limit: _____.

1.4 Example: $f(x) = \frac{x}{|x|}$, $\lim_{x \rightarrow 0^+} f(x) = \underline{\hspace{2cm}}$, and $\lim_{x \rightarrow 0^-} f(x) = \underline{\hspace{2cm}}$.1.5 *Theorem 6*

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits and these one-sided limits are equal:

$$\underline{\hspace{2cm}} \Leftrightarrow \underline{\hspace{2cm}} \quad \text{and} \quad \underline{\hspace{2cm}}$$

 **Ex. 1** (example2, p68)

Find limits of the function at $x = 0, 1, \dots, 4$ graphed below.*sol:*

2 Precise Definitions of One-Sided Limits

2.1 Definitions: Right-Hand, Left-Hand Limits

(a) We say that $f(x)$ has **right-hand limit L at x_0** , and write

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

(b) We say that $f(x)$ has **left-hand limit L at x_0** , and write

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x


2.2 $f(x) = \sin(1/x)$ has _____ as x approaches zero from either side.

3 Limits Involving $\frac{\sin \theta}{\theta}$

3.1 Theorem 7


$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

Proof:

 Ex. 2 (example5, p70)

Evaluate (a) $\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}$, (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x}$.

sol:

 Ex. 3 (example5, p70)

Find $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t}$.

sol:

實習課練習

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2. \end{cases}$

(a) Find $\lim_{x \rightarrow 2^+} f(x)$ and $\lim_{x \rightarrow 2^-} f(x)$.

(b) Does $\lim_{x \rightarrow 2} f(x)$ exist? If so, what is it? If not, why not?

(c) Find $\lim_{x \rightarrow 4^-} f(x)$ and $\lim_{x \rightarrow 4^+} f(x)$.

(d) Does $\lim_{x \rightarrow 4} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0. \end{cases}$

(a) Does $\lim_{x \rightarrow 0^+} f(x)$ exist? If so, what is it? If not, why not?

(b) Does $\lim_{x \rightarrow 0^-} f(x)$ exist? If so, what is it? If not, why not?

(c) Does $\lim_{x \rightarrow 0} f(x)$ exist? If so, what is it? If not, why not?

11. Find the limit: $\lim_{x \rightarrow -0.5^-} \sqrt{\frac{x+2}{x+1}}$.

15. Find the limit: $\lim_{h \rightarrow 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$.

18. Find the limit: (a) $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x-1)}{|x-1|}$, (b) $\lim_{x \rightarrow 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|}$.

19. Find the limit: (a) $\lim_{\theta \rightarrow 3^+} \frac{[\theta]}{\theta}$, (b) $\lim_{\theta \rightarrow 3^-} \frac{[\theta]}{\theta}$. (note: $[x]$ is the greatest integer of x)

20. Find the limit: (a) $\lim_{t \rightarrow 4^+} (t - [t])$, (b) $\lim_{t \rightarrow 4^-} (t - [t])$. (note: $[x]$ is the greatest integer of x)

25. Find the limit: $\lim_{x \rightarrow 0} \frac{\tan 2x}{x}$.

29. $\lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$.

39. $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x}$.

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2.5 Continuity

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1 Continuity at a Point

1.1 Definition: Continuity at a Point

Interior point: A function $y = f(x)$ is continuous at an _____ c of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is continuous at a _____ a or is continuous at a _____ b of its domain if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b).$$

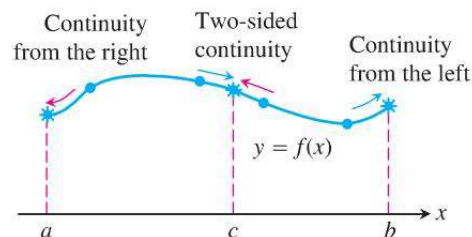


FIGURE 2.36 Continuity at points a , b , and c .

1.2 If a function is **not** continuous at a point c : f is _____ at c , and c is a point of _____ of f .

1.3 f is right-continuous (continuous from the right) at a point $x = c$ in its domain if

$$\lim_{x \rightarrow c^+} f(x) = f(c).$$

1.4 f is left-continuous (continuous from the left) at a point $x = c$ in its domain if _____.

1.5 f is continuous at a **left** endpoint a of its domain if it is _____ at a .

1.6 f is continuous at a **right** endpoint b of its domain if it is _____ at b .

1.7 f is continuous at an **interior** point c of its domain if and only if it is both _____ and _____ at c .

1.8 *Continuity Test*


A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

(a) _____ : c lies in the domain of f .

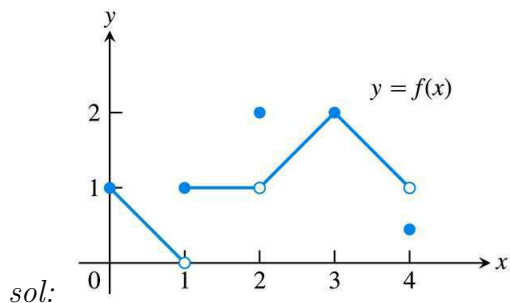
(b) _____ : f has a limit as $x \rightarrow c$.

(c) _____ : the limit equals the function value.

(a) To define continuity at a point in a function's domain, we need to define continuity at an _____ (which involves a two-sided limit) and continuity at an _____ (which involves a one-sided limit).

 **Ex. 1** (example1, p74)

Find the points at which the function f is continuous and the points at which f is discontinuous. (圖形如下)



 **Ex. 2** (example4, p75)

The function $y = [x]$ is discontinuous at every integer because the left-hand and right-hand limits are not equal as $x \rightarrow n$:

$$\lim_{x \rightarrow n^-} [x] = \underline{\hspace{2cm}} \quad \lim_{x \rightarrow n^+} [x] = \underline{\hspace{2cm}}$$

Since $[n] = n$, the greatest integer function is right-continuous at every integer n .

2 Continuous Function

2.1 A function is continuous on an interval if and only if it is _____ of the interval.

2.2 A _____ is one that is continuous at every point of its domain.

2.3 $y = 1/x$ is not continuous on $[-1, 1]$, but it is continuous over its domain _____.

2.4 *Theorem 8: Properties of Continuous Functions*

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

(a) *Sums* : _____

Prove the sum property:

(b) *Differences* : _____

(c) *Constant multiples* : _____

(d) *Products* : _____

(e) *Quotients* : _____

f *Powers* : _____ provided it is defined on an open interval, r, s : integers

(a) *Roots* : _____, provided it is defined on an open interval containing c , where n is a positive integer.

(g) Every polynomial $P(x) =$ _____ is continuous because _____

2.5 *Theorem 9: Composite of Continuous Functions*

If f is continuous at c and g is continuous at $f(c)$, then the composite _____ is continuous at c .

2.6 *Theorem 10: Composite of Continuous Functions*

If g is continuous at b and $\lim_{x \rightarrow c} f(x) = b$, then

$$\lim_{x \rightarrow c} g(f(x)) = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

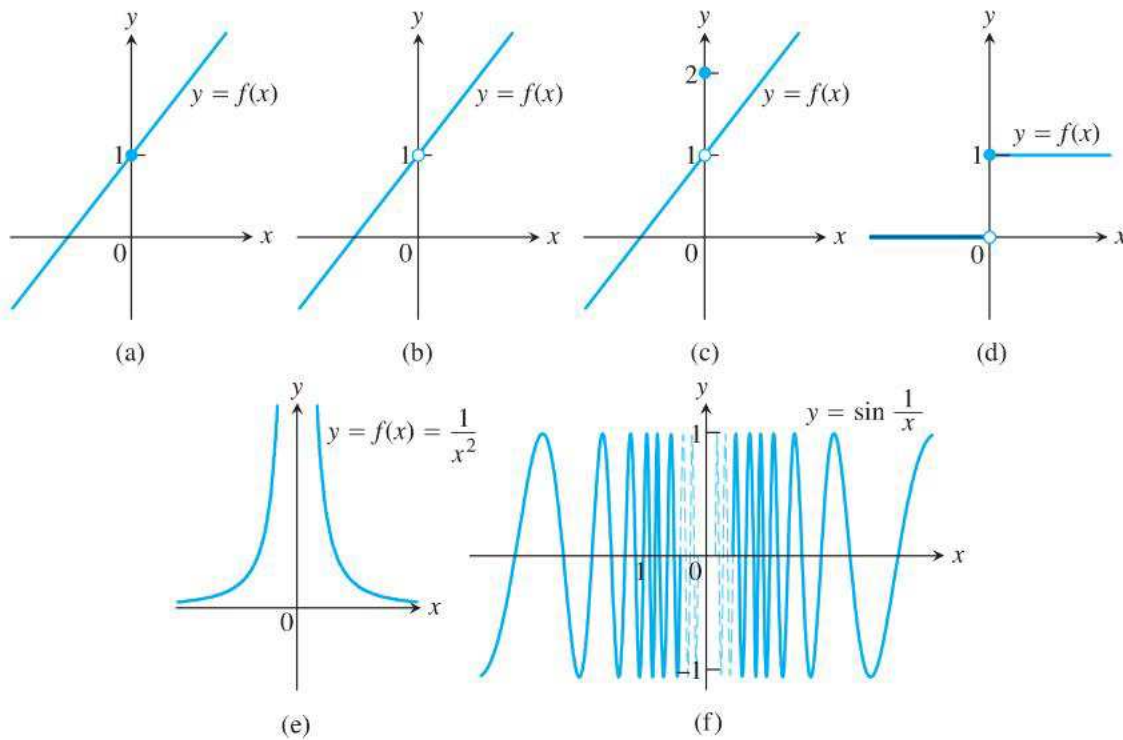


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

3 Continuous Extension to a Point


3.1 A function may have a limit even at a point where it is _____.

3.2 If $f(c)$ is not defined, but _____ exists, we can define a new function

$F(x)$ by the rule

3.3 The function F is continuous at $x = c$. It is called the _____ of f to _____.

3.4 For rational functions f , continuous extensions are usually found by _____.

 **Ex. 3** (example10, p79)

Show that $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$, $x \neq 2$ has a continuous extension to $x = 2$, and find that that extension.

sol:


4 Intermediate Value Theorem for Continuous Functions

4.1 Theorem 11: The Intermediate Value Theorem for Continuous Functions

If f is continuous function on a closed interval $[a, b]$ and if y_0 is any value between _____ and _____, then _____ for some c in $[a, b]$.


圖示如下:

4.2 The Intermediate Value Theorem says that any _____ crossing the y -axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

 **Ex. 4** (example11, p81)

Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

sol:

 Ex. 5 (example12, p81)

Use the Intermediate value Theorem to prove that the equation $\sqrt{2x+5} = 4-x^2$ has a solution.

sol:

實習課練習 (EXERCISE 2.5)

13-25. At what points are the functions in the following continuous?

13. $y = \frac{1}{x-2} - 3x.$

18. $y = \frac{1}{|x|+1} - \frac{x^2}{2}.$

24. $y = \frac{\sqrt{x^4+1}}{1+\sin^2 x}.$

25. $y = \sqrt{2x+3}.$

37. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

44. For what value of b is

$$g(x) = \begin{cases} \frac{x-b}{b+1} & x < 0, \\ x^2 + b & x > 0 \end{cases}$$

continuous at every x .

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2.6 Limits Involving Infinity; Asymptotes of Graphs

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1 Finite Limits as $x \rightarrow \pm\infty$

1.1 Definitions: Limit as x approaches ∞ or $-\infty$

(a) We say that $f(x)$ has the **limit L as x approaches infinity** and write

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x _____.

(b) We say that $f(x)$ has the **limit L as x approaches minus infinity** and write _____ if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x _____.

1.2 Theorem 12: Limit Laws as $x \rightarrow \pm\infty$

If L, M , and k are real numbers and $\lim_{x \rightarrow \pm\infty} f(x) = L$ and $\lim_{x \rightarrow \pm\infty} g(x) = M$, then

(a) sum rule: _____.


(b) difference rule: _____.

(c) product rule: _____.

(d) constant multiple rule: _____.


(e) quotient rule: _____.

(f) power rule: if r and s are integers with no common factor and $s \neq 0$, then _____, provided that $L^{r/s}$ is a real number.

 **Ex. 1** (example1, p85)


Show that (a) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$, (b) $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

sol:

 **Ex. 2** (example2, p85)

Evaluate (a) $\lim_{x \rightarrow \infty} (5 + \frac{1}{x})$, (b) $\lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2}$.

sol:

 **Ex. 3** (example3, p86)

Evaluate (a) $\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2}$, (b) $\lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1}$.

sol:

2 Horizontal Asymptotes


2.1 Definition: Horizontal Asymptote

A line $y = b$ is a _____ of the graph of a function $y = f(x)$ if either _____ or _____ .

2.2 Example 1: $f(x) = \frac{1}{x}$, the _____ is an asymptote of the curve on the right because _____ and on the left because _____ .

2.3 Example 2: $f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$ has the line _____ as a horizontal asymptote on both right and the left because _____ and _____ .

2.4 The Sandwich Theorem also holds for limits as $x \rightarrow \pm\infty$.

 **Ex. 4** (example4, p87)

Find the horizontal asymptotes of the graph of $f(x) = \frac{x^3 - 2}{|x|^3 + 1}$.

sol:

 **Ex. 5** (example5, p87)

(a) Find $\lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right)$, (b) Find $\lim_{x \rightarrow \pm\infty} x \sin\left(\frac{1}{x}\right)$.

sol:

 Ex. 6 (example6, p87)

Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

sol:

 Ex. 7 (example7, p88)

Find $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16})$.

sol:

3 Oblique Asymptotes

- 3.1 If the _____ of a rational function is greater than the _____ the graph has an _____ or _____.
- 3.2 We find an equation for the asymptote by dividing numerator by denominator to express f as a _____ plus a _____ that goes to zero as _____.

 Ex. 8 (example8, p88)

Find the oblique asymptote of the graph of $f(x) = \frac{x^2 - 3}{2x - 4}$.

sol:

4 Infinite Limits


4.1 The limit does not exist: $\lim_{x \rightarrow 0^-} \frac{1}{x} = \underline{\hspace{2cm}}$, $\lim_{x \rightarrow 0^+} \frac{1}{x} = \underline{\hspace{2cm}}$.

4.2 The limit $\lim_{x \rightarrow \infty} \frac{1}{x} = \underline{\hspace{2cm}}$ does not exist.

 Ex. 9 (example9, p89)

Find $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$ and $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$.

sol:

 Ex. 10 (example11, p90)

$$(a) \lim_{x \rightarrow 2} \frac{(x-2)^2}{x^2-4} =$$

$$(b) \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} =$$

$$(c) \lim_{x \rightarrow 2^+} \frac{x-3}{x^2-4} =$$

$$(d) \lim_{x \rightarrow 2^-} \frac{x-3}{x^2-4} =$$

$$(e) \lim_{x \rightarrow 2} \frac{x-3}{x^2-4} =$$

$$(f) \lim_{x \rightarrow 2} \frac{2-x}{(x-2)^3} =$$

5 Precise Definitions of Infinite Limits

5.1 Definitions: Infinity, Negative Infinity as Limits


(a) We say that $f(x)$ approaches _____ as x approaches x_0 , and write _____ if for every positive number B there exists a corresponding $\delta > 0$ such that for all x _____ \Rightarrow _____

(b) We say that $f(x)$ approaches _____ as x approaches x_0 , and write _____ if for every negative number $-B$ there exists a corresponding $\delta > 0$ such that for all x _____ \Rightarrow _____

6 Vertical Asymptotes


6.1 Definitions: Vertical Asymptote

A line _____ is a vertical asymptote of the graph of a function $y = f(x)$ if either _____ or _____

 **Ex. 11** (example13, p91)

Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$.

sol:

 **Ex. 12** (example14, p92)

Find the horizontal and vertical asymptotes of the graph of $f(x) = -\frac{8}{x^2 - 4}$.

sol:

實習課練習 (EXERCISE 2.6)

10. Find the limit $\lim_{\theta \rightarrow -\infty} \frac{\cos \theta}{3\theta}$.
16. Find the limit (a) as $x \rightarrow \infty$ and (b) as $x \rightarrow -\infty$: $f(x) = \frac{3x + 7}{x^2 - 2}$.
25. Find the limit: $\lim_{x \rightarrow -\infty} \left(\frac{1 - x^3}{x^2 + 7x} \right)^5$.
28. Find the limit: $\lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$.
41. Find the limit: $\lim_{x \rightarrow -8^+} \frac{2x}{x + 8}$.
44. Find the limit: $\lim_{x \rightarrow 0} \frac{-1}{x^2(x + 1)}$.
53. Find the limits: $\lim_{x \rightarrow 2^+} \frac{1}{x^2 - 4}$ as
(a) $x \rightarrow 2^+$, (b) $x \rightarrow 2^-$, (c) $x \rightarrow -2^+$, and (d) $x \rightarrow -2^-$.
55. Find the limits: $\lim_{x \rightarrow 0^+} \left(\frac{x^2}{2} - \frac{1}{x} \right)$ as
(a) $x \rightarrow 0^+$, (b) $x \rightarrow 0^-$, (c) $x \rightarrow \sqrt[3]{2}$, and (d) $x \rightarrow -1$.
61. Find the limits: $\lim_{x \rightarrow 1^+} \left(\frac{1}{x^{2/3}} + \frac{2}{(x - 1)^{2/3}} \right)$ as
(a) $x \rightarrow 0^+$, (b) $x \rightarrow 0^-$, (c) $x \rightarrow 1^+$, and (d) $x \rightarrow 1^-$.
80. Find the limit: $\lim_{x \rightarrow \infty} (\sqrt{x + 9} - \sqrt{x + 4})$.

THOMAS' CALCULUS (12/E)

3.1 Tangents and the Derivative at a Point

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>**1 Finding a Tangent to the Graph of a Function**1.1 *Definition: Slope, Tangent Line*

The **slope** of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad (\text{provided the limit exists})$$

The **tangent line** to the curve at P is the line through P with this slope.

1.2 *Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)*

(a) Calculate $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$ and $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$.

(b) Calculate the slope $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$.

(c) If the limit exists, find the tangent line as $y - y_0 = m(x - x_0)$. (point-slope equation)

 **Ex. 1** (example1, p103)

(a) Find the slope of the curve $y = 1/x$ at $x = a \neq 0$. What is the slope at the point $x = -1$.

(b) Where does the slope equal $-1/4$.

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

sol:

2 Rates of Change: Derivative at a Point

2.1 The difference quotient of f at x_0 with increment h : _____ .

2.2 If the difference quotient has a limit as h approaches _____, that limit is called the _____ of f at x_0 .

2.3 Definition

The derivative of a function f at a point x_0 , denoted $f'(x_0)$, is _____ provided this limit exists. _____

2.4 Summary

All the following refer to the same thing.

(a) The _____ of $y = f(x)$ at $x = x_0$.

(b) The _____ to the curve $y = f(x)$ at $x = x_0$.

(c) The _____ of $f(x)$ with respect to x at $x = x_0$.

(d) The _____, $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

(e) The _____ of f at $x = x_0$.

實習課練習 (EXERCISE 3.1)

Find an equation for the line tangent to the graph at the given point.

8. $y = \frac{1}{x^2}$, $(-1, 1)$.

11. $f(x) = x^2 + 1$, $(2, 5)$.

13. $g(x) = \frac{x}{x-2}$, $(3, 3)$.

18. $f(x) = \sqrt{x+1}$, $(8, 3)$.

THOMAS' CALCULUS (12/E)

3.2 The Derivative as a Function

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Calculating Derivatives from the Definition

1.1 The slope of a curve $y = f(x)$ at the point where $x = x_0$:

this limit is called the _____ of f at x_0 .

1.2 *Definition: Derivative Function*

The _____ of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

provided the limit exists. _____

1.3 Alternative Formula for the Derivative

1.4 If _____ exist at a particular x , we say that f is _____ (has a derivative) at x .

1.5 If f' exists at _____ in the domain of f , we call f _____.

1.6 The process of calculating a derivative: _____.


1.7 $f'(x) =$ _____.

1.8 y : the _____ variable. x : the _____ variable.

2.2 A function $y = f(x)$ is _____ on a _____ $[a, b]$ if it is differentiable on the interior (a, b) and if the limits exist at the endpoints.


2.3 Right-hand derivative at a :

2.4 Left-hand derivative at b :

 **Ex. 3** (example4, p110)

Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

sol:

 **Ex. 4** (example5, p110)

Show that the function $y = \sqrt{x}$ is not differentiable at $x = 0$.

sol:

3 Differentiable Functions are Continuous

3.1 A function has a derivative at a point x_0 if the _____ through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a _____ as Q approaches P .

3.2 When does a function not have a derivative at a point?

- (a) a _____, where the one-sided derivatives differ.
- (b) a _____, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.
- (c) a _____, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).
- (d) a _____.

3.3 A function is continuous at every point where it has a _____.

3.4 *Theorem 1: Differentiability Implies Continuity*

If f has a _____ at $x = c$, then f is _____ at $x = c$.

Proof:

3.5 A function **need not** have a derivative at a point where it is continuous.

實習課練習 (EXERCISE 3.2)

Using the definition, calculate the derivative of the function in Exercise 2 and 4. Then find the values of the derivatives as specified.

2. $F(x) = (x - 1)^2 + 1$; $F'(-1)$, $F'(0)$, $F'(2)$.

4. $k(z) = \frac{1 - z}{2z}$; $k'(-1)$, $k'(1)$, $k'(\sqrt{2})$.

13. Differentiate the functions and find the slope of the tangent line at the given value of the independent variable. $f(x) = x + \frac{9}{x}$, $x = -3$.

17. Differentiate the function and find the equation of the tangent line at the indicated point on the graph of the function. $y = f(x) = \frac{8}{\sqrt{x-2}}$, $(x, y) = (6, 4)$.

26. Use the alternative formula for derivative to find the derivative of the function: $g(x) = 1 + \sqrt{x}$.

41. determine if the piecewise defined function is differentiable at the origin.

$$f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$$

61. Derivative of $y = |x|$. Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

THOMAS' CALCULUS (12/E)

3.3 Differentiation Rules

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授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Powers, Multiples, Sums, and Differences

1.1 *RULE 1: Derivative of a Constant Function*

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

Proof:

1.2 *RULE 2: Power Rule for Positive Integers*

If n is a positive integer, then

$$\frac{d}{dx} x^n = \underline{\hspace{2cm}}$$

Proof:

1.3 *Power Rule (General Version)*

If n is any real number, then

$$\frac{d}{dx} x^n = \underline{\hspace{2cm}}$$

for all x where the power x^n and x^{n-1} are defined.

1.4 *RULE 3: Constant Multiple Rule*


If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = \underline{\hspace{2cm}}$$

1.5 *RULE 4: Derivative Sum Rule*


If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \underline{\hspace{2cm}}$$

 **Ex. 1** (example1, p117)

Differentiate the following powers of x . (a) x^3 , (b) $x^{2/3}$, (c) $x^{\sqrt{2}}$, (d) $\frac{1}{x^4}$, (e) $x^{-4/3}$, (f) $\sqrt{x^{2+\pi}}$.

sol:

 **Ex. 2** (example3, p118)

Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

sol:

2 Products

2.1 *RULE 5: Derivative Product Rule*

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = \underline{\hspace{2cm}}$$

2.2 $\frac{d}{dx}[f(x)g(x)] = \underline{\hspace{2cm}}$

Proof:

 **Ex. 3** (example5, p119)

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

sol:

3 Quotients


3.1 *RULE 6: Derivative Quotient Rule*

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \underline{\hspace{2cm}}$$

3.2 $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \underline{\hspace{2cm}}$

Proof:

 **Ex. 4** (example6, p120)

Find the derivative of $y = \frac{t^2 - 1}{t^2 + 1}$.


sol:

4 Second- and Higher-Order Derivatives

4.1 The function _____ is called the _____ of f because it is the derivative of the first derivative.

$$f''(x) = \underline{\hspace{15em}}$$

4.2 $f^{(n)}(x)$: the _____ of y with respect to x for any positive integer n .

 **Ex. 5** (example8, p122)

Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

sol:

實習課練習 (EXERCISE 3.3)

5. Find the first and second derivatives of $y = \frac{4x^3}{3} - x$.
8. Find the first and second derivatives of $s = -2t^{-1} + \frac{4}{t^2}$.
15. Find y' of $y = (x^2 + 1)(x + 5 + \frac{1}{x})$.
21. Find the derivative of $v = (1 - t)(1 + t^2)^{-1}$.
24. Find the derivative of $u = \frac{5x + 1}{2\sqrt{x}}$.
28. Find the derivative of $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$.
34. Find the first and second derivatives of $s = \frac{t^2 + 5t - 1}{t^2}$.
38. Find the first and second derivatives of $w = (z + 1)(z - 1)(z^2 + 1)$.
- 53(a). Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$
55. Evaluate $\lim_{x \rightarrow 1} \frac{x^{50} - 1}{x - 1}$.

THOMAS' CALCULUS (12/E)

3.5 Derivatives of Trigonometric Functions

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Derivative of the Sine, Cosine Function

1.1 *The derivative of the sine function is the cosine function*


$$\frac{d}{dx}(\sin x) = \underline{\hspace{2cm}}$$

Proof:

1.2 *The derivative of the cosine function is the sine function*

$$\frac{d}{dx}(\cos x) = \underline{\hspace{2cm}}$$

Proof:

 **Ex. 1** (example1, p136)

Find $\frac{dy}{dx}$ if (a) $y = x^2 - \sin x$, (b) $y = x^2 \sin x$, (c) $y = \frac{\sin x}{x}$.

sol:

 **Ex. 2** (example2, p137)


Find $\frac{dy}{dx}$ if (a) $y = 5x + \cos x$, (b) $y = \sin x \cos x$, (c) $y = \frac{\cos x}{1 - \sin x}$.

sol:

2 Derivatives of the Other Basic Tri. Functions


2.1 Derivatives of the Other Trigonometric Functions

| |
|---|
| <p>(a) $\frac{d}{dx}(\tan x) = \underline{\hspace{2cm}}$. (b) $\frac{d}{dx}(\cot x) = \underline{\hspace{2cm}}$.</p> <p>(c) $\frac{d}{dx}(\sec x) = \underline{\hspace{2cm}}$. (d) $\frac{d}{dx}(\csc x) = \underline{\hspace{2cm}}$.</p> |
|---|

 Ex. 3 (example5, p139)

Find $\frac{d}{dx}(\tan x)$

sol:

 Ex. 4 (example6, p139)

Find y'' if $y = \sec x$.

sol:

實習課練習 (EXERCISE 3.5)

2. Find $\frac{dy}{dx}$ if $y = \frac{3}{x} + \sin x$.

7. Find $\frac{dy}{dx}$ if $y = f(x) = \sin x \tan x$.

9. Find $\frac{dy}{dx}$ if $y = (\sec x + \tan x)(\sec x - \tan x)$.

12. Find $\frac{dy}{dx}$ if $y = \frac{\cos x}{1 + \sin x}$.

21. Find $\frac{ds}{dt}$ if $s = \frac{1 + \csc t}{1 - \csc t}$.

26. Find $\frac{dr}{d\theta}$ if $r = (1 + \sec \theta) \sin \theta$.

29. Find $\frac{dp}{dq}$ if $p = \frac{\sin q + \cos q}{\cos q}$.

34. Find $y^{(4)} = \frac{d^4y}{dx^4}$ if (a) $y = -2 \sin x$, (b) $y = 9 \cos x$.

THOMAS' CALCULUS (12/E)

3.6 The Chain Rule

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Derivative of a Composite Function

1.1 Theorem 3: The Chain Rule

If $f(u)$ is differentiable at the point $u = \underline{\hspace{2cm}}$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and


$$(f \circ g)'(x) = \underline{\hspace{2cm}}.$$

1.2 If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \underline{\hspace{2cm}}$, where dy/du is evaluated at $u = g(x)$.

1.3 Think of the derivative as a rate of change: if $y = f(u)$ changes half as fast as u and $u = g(x)$ changes three times as fast as x , then we expect y to change $\underline{\hspace{2cm}}$ times as fast as x .


1.4 Repeated Use of the Chain Rule

$$\frac{dy}{dx} = \underline{\hspace{2cm}}$$

 **Ex. 1** (example1, p142)


The function $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculate $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{du} \cdot \frac{du}{dx}$. Use the Chain Rule to calculate $\frac{dy}{dx}$.

sol:

 Ex. 2 (example3, p144)

Differentiate $\sin(x^2 + x)$ with respect to x .

sol:

 Ex. 3 (example4, p145)

Find the derivative of $g(t) = \tan(5 - \sin 2t)$


sol:

2 The Power Chain Rule

2.1 *The Chain Rule with Powers of a Function*


If n is a positive or negative integer, $f(u) = u^n$, and u is a differentiable function of x , then

$$\frac{d}{dx}u^n = \underline{\hspace{2cm}}$$

 Ex. 4 (example5, p145)

(a) $\frac{d}{dx}(5x^3 - x^4)^7$, (b) $\frac{d}{dx}\left(\frac{1}{3x-2}\right)$, (c) $\frac{d}{dx}(\sin^5 x)$

sol:

 Ex. 5 (example6, p146)

$\frac{d}{dx}(|x|)$.

sol:

實習課練習 (EXERCISE 3.6)

1. Find $\frac{dy}{dx}$ if $y = 6u - 9$, $u = (1/2)x^4$.
6. Find $\frac{dy}{dx}$ if $y = \sin u$, $u = x - \cos x$.
9. Find $\frac{dy}{dx}$ if $y = (2x + 1)^5$.
16. Find $\frac{dy}{dx}$ if $y = \cot(\pi - \frac{1}{x})$.
20. Find the derivative of $q = \sqrt{2r - r^2}$.
35. Find the derivative of $f(\theta) = (\frac{\sin \theta}{1 + \cos \theta})^2$.
41. Find dy/dt of $y = \sin^2(\pi t - 2)$.
53. Find dy/dt of $y = \sqrt{1 + \cos(t^2)}$.
60. Find y'' of $y = (1 - \sqrt{x})^{-1}$.
67. Find the value of $(f \circ g)'$ at the given value of x .
 $f(u) = \cot \frac{\pi u}{10}$, $u = g(x) = 5\sqrt{x}$, $x = 1$.
70. Find the value of $(f \circ g)'$ at the given value of x .
 $f(u) = (\frac{u-1}{u+1})^2$, $u = g(x) = \frac{1}{x^2} - 1$, $x = -1$.
- 81(a). Find the tangent to the curve $y = 2 \tan(\pi x/4)$ at $x = 1$.

THOMAS' CALCULUS (12/E)

3.7 Implicit Differentiation

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>**1 Implicit Differentiation**

1.1 An equation of the form _____ : express y explicitly in terms of the variable x .

1.2 The parametric equation: _____ and _____ .

1.3 An _____ between the variables x and y :
e.g., $x^2 + y^2 = 25$, $y^2 - x = 0$, or $x^3 + y^3 - 9xy = 0$.

1.4 *Implicit Differentiation*

(a) Differentiate _____ of the equation with respect to x , treating y as a differentiable function of x .


(b) Collect the term with _____ on one side of the equation.

(c) Solve for _____ .

 **Ex. 1** (example1, p150)


Find dy/dx if $y^2 = x$.

sol:

 Ex. 2 (example2, p150)


Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

sol:

 Ex. 3 (example3, p151)

Find dy/dx if $y^2 = x^2 + \sin xy$.

sol:

 Ex. 4 (example4, p152)

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

sol:

實習課練習 (EXERCISE 3.7)

1. Use implicit differentiation to find dy/dx if $x^2y + xy^2 = 6$.
 5. Use implicit differentiation to find dy/dx if $x^2(x - y)^2 = x^2 - y^2$.
 8. Use implicit differentiation to find dy/dx if $x^3 = \frac{2x - y}{x + 3y}$.
 13. Use implicit differentiation to find dy/dx if $y \sin\left(\frac{1}{y}\right) = 1 - xy$.
 24. Use implicit differentiation to find dy/dx and d^2y/dx^2 if $xy + y^2 = 1$.
- 38(a). Find the line that is tangent to the curve at the given points. $x^2 \cos^2 y - \sin y = 0$, $(0, \pi)$.

THOMAS' CALCULUS (12/E)

3.9 Linearization and Differentials

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Linearization

1.1 Definitions: Linearization, Standard Linear Approximation


(a) If f is differentiable at $x = a$, then the approximating function

$L(x) = \underline{\hspace{4cm}}$

is the $\underline{\hspace{4cm}}$ of f at a .


(b) The approximation $f(x) \approx L(x)$ of f by L is the $\underline{\hspace{4cm}}$
 $\underline{\hspace{4cm}}$ of f at a .

(c) The point $x = a$ is the $\underline{\hspace{4cm}}$ of the approximation.

 **Ex. 1** (example1, p165)


Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$.

sol:

 **Ex. 2** (example2, p166)

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

sol:

 **Ex. 3** (example3, p166)

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$.

sol:


2 Differentials

1. *Definition: Differential*

If $y = f(x)$ be a differentiable function. The _____ dx is an independent variable. The differential dy is


$$dy = \underline{\hspace{2cm}}$$

2. dy is always a _____ variable. It depends on both x and dx .

 **Ex. 4** (example4, p167)

(a) Find dy if $y = x^5 + 37x$, (b) find the value of dy when $x = 1$ and $dx = 0.2$.

sol:

 **Ex. 5** (example5, p168)

(a) $d(\tan 2x)$, (b) $d\left(\frac{x}{x+1}\right)$.

sol:

實習課練習 (EXERCISE 3.9)

1. Find the linearization $L(x)$ of $f(x) = x^3 - 2x + 3$ at $x = 2$.
4. Find the linearization $L(x)$ of $f(x) = \sqrt[3]{x}$ at $x = -8$.
- 6(a). Find the linearization $L(x)$ of $f(x) = \sin x$ at $x = 0$.
17. Find dy if $y = x^3 - 3\sqrt{x}$.
20. Find dy if $y = \frac{2\sqrt{x}}{3(1+\sqrt{x})}$.
21. Find dy if $2y^{3/2} + xy - x = 0$.
28. Find dy if $y = 2 \cot\left(\frac{1}{\sqrt{x}}\right)$.
30. $f(x) = 2x^2 + 4x - 3$, $x_0 = -1$, $dx = 0.1$. Each function $f(x)$ changes values when x changes from x_0 to $x_0 + dx$. Find
 - (a) the change $\Delta f = f(x_0 + dx) - f(x_0)$;
 - (b) the value of the estimate $df = f'(x_0) dx$; and
 - (c) the approximation error $|\Delta f - df|$.

THOMAS' CALCULUS (12/E)

4.1 Extreme Values of Functions

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Global (Absolute) Extreme Values

1.1 Definitions: Absolute Maximum, Absolute Minimum

Let f be a function with domain D . Then

(a) f has an _____ on D at a point c if _____
for all x in D , and

(b) an _____ on D at c if _____ for all x in
 D .

1.2 Absolute maximum and minimum values are called _____ or
_____.

1.3 Theorem 1: The Extreme Value Theorem

If f is _____ on a _____ $[a, b]$, then f attains both an
absolute maximum value M and an absolute minimum value m in $[a, b]$. That is,
there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = \underline{\hspace{2cm}}$, $f(x_2) = \underline{\hspace{2cm}}$, and
_____ for every other x in $[a, b]$.

1.4 Some possibilities for a continuous function's maximum and minimum on a closed
interval $[a, b]$. (圖示如下)

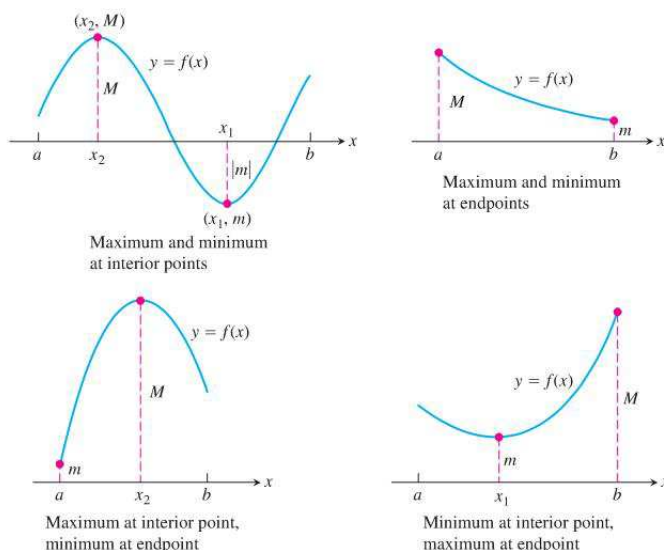


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

 **Ex. 1** (example1, p185)

Functions with the same defining rule can have different extrema, depending on the domain.

| Function rule | Domain D | Absolute extrema on D |
|---------------|---------------------|-------------------------|
| (a) $y = x^2$ | $(-\infty, \infty)$ | _____. |
| (b) $y = x^2$ | $[0, 2]$ | _____. |
| (c) $y = x^2$ | $(0, 2]$ | _____. |
| (d) $y = x^2$ | $(0, 2)$ | _____. |

2 Local (Relative) Extreme Values

2.1 Definitions: Local Maximum, Local Minimum

- (a) A function f has a _____ at an _____ c of its domain if _____ for all x in some open interval containing c .
- (b) A function f has a _____ at an _____ c of its domain if _____ for all x in some open interval containing c .

2.2 Local extrema are also called _____.

2.3 An absolute maximum is also a _____.

2.4 How to classify maxima and minima: (圖示如下)

3 Finding Extrema

3.1 *Theorem 2: The First Derivative Theorem for Local Extreme Values*

If f has a _____ or _____ at an interior point c of its domain, and if f' is defined at c , then _____.

3.2 A function's _____ is always _____ at an interior point where the function has a _____ and the derivative is defined.

3.3 Where a function f can possibly have an extreme value (local or global) are

- (a) the interior points where _____,
- (b) the interior points where _____.
- (c) the _____ of the domain of f .


3.4 *Definition: Critical Point*

An _____ of the domain of a function f where f' is _____ or _____ is a _____ of f .

3.5 The only domain points where a function can assume extreme values are _____ and _____.

3.6 How to find the **absolute extrema** of a continuous function f on a finite closed interval.

- (a) Evaluate f at all _____ and _____.
- (b) Take the _____ and _____ of these values.

 Ex. 2 (example2, p188)

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

sol:

 Ex. 3 (example3, p189)

Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

sol:

 Ex. 4 (example4, p189)

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on $[-2, 3]$.

sol:

實習課練習 (EXERCISE 4.1)

29. Find the absolute maximum and minimum values of $g(x) = \sqrt{4 - x^2}$ on $-2 \leq x \leq 1$.

31. Find the absolute maximum and minimum values of $f(\theta) = \sin \theta$ on $-\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}$.

36. Find the absolute maximum and minimum values of $f(t) = |t - 5|$ on $4 \leq t \leq 7$.

58. Find the extreme values of $y = \frac{x + 1}{x^2 + 2x + 2}$ and where they occur.

64. Find the derivative at each critical point and determine the local extreme values.

$$y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \geq 0 \end{cases}$$

68. Determine all extrema of $f(x) = |x^3 - 9x|$.

THOMAS' CALCULUS (12/E)

4.2 The Mean Value Theorem

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Rolle's Theorem

1.1 Theorem 3: Rolle's Theorem

Suppose that $y = f(x)$ is _____ at every point of the closed interval $[a, b]$ and _____ at every point of its interior (a, b) . If _____, then there is at least one number c in (a, b) at which _____.

1.2 Rolle's Theorem says that a differentiable curve had at least one _____ between any two points where it crosses a _____. It may have just one, or more. (圖示如下)

2 The Mean Value Theorem

2.1 Theorem 4: The Mean Value Theorem

Suppose that $y = f(x)$ is _____ on a closed interval $[a, b]$ and _____ on the interval's interior (a, b) . Then there is at least one point c in (a, b) at which _____.


2.2 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one _____ parallel to chord AB . (圖示如下)

2.3 *A Physical Interpretation*

If we think of the number _____ as the _____ in f over $[a, b]$ and _____ as an _____, then the Mean Value Theorem says that at _____ the instantaneous change must _____ the average change over the entire interval.

2.4 *Mathematical Consequences*

- (a) If _____ at each point x of an open interval (a, b) , then _____ for all $x \in (a, b)$, where C is a constant.
- (b) If _____ at each point x in an open interval (a, b) , then there exists a constant C such that _____ for all $x \in (a, b)$. That is, _____ is a constant on (a, b) .

 **Ex. 1** (example1, p193)

Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

sol:

 **Ex. 2** (example2, p194)

Find the values of c that satisfy the Mean Value Theorem for $f(x) = x^2$ on $[0, 2]$.

sol:

實習課練習 (EXERCISE 4.2)

Find the values of c that satisfy the Mean Value Theorem for the functions and intervals.

1. $f(x) = x^2 + 2x - 1$, $[0, 1]$.

3. $f(x) = x + \frac{1}{x}$, $[1/2, 2]$.

4. $f(x) = \sqrt{x-1}$, $[1, 3]$.

14. For what values of a , m and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \leq x \leq 2 \end{cases}$$

satisfy the hypothesis of the Mean value Theorem on the interval $[0, 2]$?

THOMAS' CALCULUS (12/E)

4.3 Monotonic Functions and The First Derivative Test

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Increasing Functions and Decreasing Functions

1.1 What kinds of functions have positive derivatives or negative derivatives?

- (a) The only functions with positive derivatives are _____ functions.
 (b) The only functions with negative derivatives are _____ functions.

1.2 *Definitions: Increasing, Decreasing Function*

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I .

(a) If _____ whenever then _____ is said to be _____ on I .

(b) If _____ whenever then _____ is said to be _____ on I .

A function that is increasing or decreasing on I is called _____ on I .

1.3 **Example:**

- (a) The function $f(x) = x^2$ decreases on _____ and increases on _____.
 (b) The function $f(x) = x^2$ is monotonic on _____ and _____ but it is not monotonic on _____. (圖示如下)

1.4 *COROLLARY 3: First Derivative Test for Monotonic Functions*

Suppose that f is _____ on $[a, b]$ and _____ on (a, b) .

(a) If _____ at each point $x \in (a, b)$, then f is _____ on $[a, b]$.

(b) If _____ at each point $x \in (a, b)$, then f is _____ on $[a, b]$.

 **Ex. 1** (example1, p199)

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

sol:

2 First Derivative Test for Local Extrema

2.1 First Derivative Test for Local Extrema

Suppose that c is a _____ of a continuous function f and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

(a) If f' changes _____ at c , then f has a _____ at c ;

(b) If f' changes from _____ at c , then f has a _____ at c ;

(c) If does not change sign at c (that is, is positive on both sides of c or negative on both sides), then f has _____ at c .

2.2 A function's first derivative tells how the graph rises and falls.

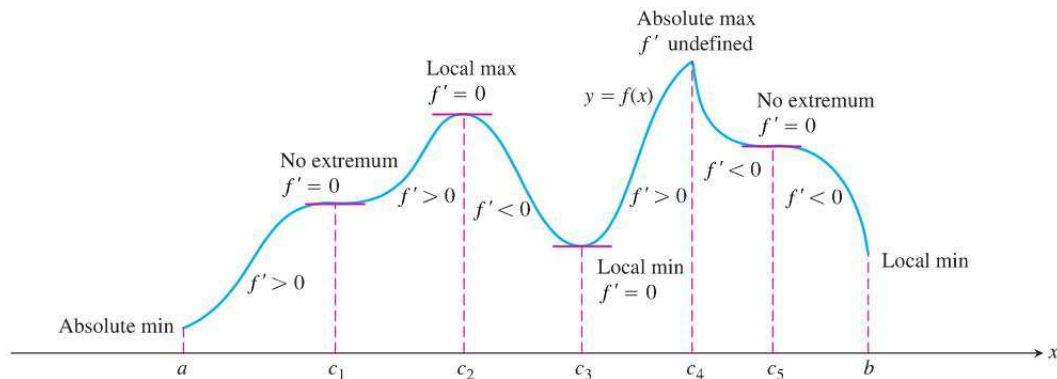



FIGURE 4.21 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

 **Ex. 2** (example2, p201)

Find the critical points of $f(x) = x^{1/3}(x - 4)$. Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

sol:

實習課練習 (EXERCISE 4.3)

Answer the following questions about the functions whose derivatives are given in Exercise 1-14.

- (a) What are the critical points of f ?
- (b) On what intervals is f increasing or decreasing?
- (c) At what points, if any, does f assume local maximum and minimum values?

4. $f'(x) = (x - 1)^2(x + 2)^2$.

11. $f'(x) = x^{-1/3}(x + 2)$.

In Exercise 15-40.

- (a) Find the open intervals on which the function is increasing and decreasing.
- (b) Identify the function's local extreme values, if any, saying where they occur.

23. $f(\theta) = 3\theta^2 - 4\theta^3$.

34. $g(x) = x^2\sqrt{5 - x}$.

36. $f(x) = \frac{x^3}{3x^2 + 1}$.

39. $h(x) = x^{1/3}(x^2 - 4)$.

69. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

THOMAS' CALCULUS (12/E)

4.4 Concavity and Curve Sketching

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Concavity

1.1 Definitions: Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$

(a) is _____ on an open interval I if _____ is _____ on I

(b) is _____ on an open interval I if _____ is _____ on I .

1.2 The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I .

(a) If _____ on I , the graph of f over I is _____.

(b) If _____ on I , the graph of f over I is _____.

1.3 Example:

(a) The graph of $f(x) = x^3$ is concave down on _____ and concave up on _____.

(b) The curve $y = x^2$ is concave up on _____ because its second derivative $y'' = 2$ is always positive. (圖示如下)

 **Ex. 1** (example2, p204)

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

sol:

2 Points of Inflection

2.1 Definitions: Point of Inflection

A point where the graph of a function has a _____ and where the _____ is a point of _____.

2.2 Example:

- (a) The curve $y = 3 + \sin x$ changes concavity at the point _____. We call _____ point of inflection of the curve.
- (b) The curve $y = x^4$ has _____ at $x = 0$. Even though $y'' = 12x^2$ is zero there, it does not change sign.
- (c) The curve $y = x^{1/3}$ has a point of inflection at $x = 0$, but does not exist there.

$$y'' = \underline{\hspace{2cm}} = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

3 Second Derivative Test for Local Extrema


3.1 Theorem: Second Derivative Test for Local Extrema

Suppose f is continuous on an open interval that contains

- (a) If _____ and _____ then f has a _____ at $x = c$.
- (b) If _____ and _____ then f has a _____ at $x = c$.
- (c) If _____ and _____ then the test fails. The function f may have a local maximum, a local minimum, or neither.

3.2 Strategy for Graphing $y = f(x)$


- (a) Identify the _____ of f and any _____ the curve may have.
- (b) Find _____ and _____.
- (c) Find the _____ of f and identify the function's behavior at each one.
- (d) Find where the curve is _____ and where it is _____.
- (e) Find the _____, if any occur, and determine the _____ of the curve.
- (f) Identify any _____.
- (g) Plot _____, such as the intercepts and the points found in Steps (c)-(e), and sketch the curve.

 **Ex. 2** (example7, p207)

Sketch a graph of the function $f(x) = x^4 - 4x^3 = 10$ using the following steps.


- (a) Identify where the extrema of f occur.
- (b) Find the intervals on which f is increasing and the intervals on which f is decreasing.
- (c) Find where the graph of f is concave up and where it is concave down.
- (d) Sketch the general shape of the graph for f .
- (e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

sol:

 Ex. 3 (example8, p208)

Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

sol:

 Ex. 4 (example9, p209)

Sketch the graph of $f(x) = \frac{x^2+4}{2x}$.

sol:

實習課練習 (EXERCISE 4.4)

Graph the equations. Include the coordinates of any local extreme points and inflection points.

21. $y = x^4(x - 5)$.

23. $y = x + \sin x, 0 \leq x \leq 2\pi$.

46. $y = |x^2 - 2x|$.

83. Graph the rational function: $y = \frac{x^2}{x + 1}$.

THOMAS' CALCULUS (12/E)


4.5 Applied Optimization

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)


教學網站: <http://www.hmwu.idv.tw>

1 Solving Applied Optimization Problems

 Ex. 1 (example1, p214)

An open-top box is to be made by cutting small congruent squares from the corners of a 12cm by 12cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible.

sol:

 Ex. 2 (example3, p216)

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

sol:

實習課練習 (EXERCISE 4.5)

4. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?
13. Two sides of a triangle have lengths a and b , and the angle between them is θ . What value of θ will maximize the triangle's area?
30. Find a positive number for which the sum of it and its reciprocal and four times its square is the smallest possible.

THOMAS' CALCULUS (12/E)

4.6 Newton's Method

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)


教學網站: <http://www.hmwu.idv.tw>

1 Procedure for Newton's Method

- 1.1 Newton's Method (also called _____ Method) is a technique to _____ the solution to an equation _____.
- 1.2 Essentially it uses _____ in place of the graph of $y = f(x)$ near the points where _____.
- 1.3 A value of x where _____ is a _____ of the function f and a _____ of the equation $f(x) = 0$.
- 1.4 *Procedure for Newton's Method*

- (a) Guess a _____ to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
- (b) Use the first approximation to get a second, the second to get a third, and so on, using the formula
- $$x_{n+1} = \frac{x_n - f(x_n)/f'(x_n)}{1 - f'(x_n)/f''(x_n)}, \quad \text{if } f'(x_n) \neq 0.$$


圖示說明如下:

 Ex. 1 (example1, p226)

Find the positive root of the equation $f(x) = x^2 - 2 = 0$.

(We find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.)

sol:

 Ex. 2 (example2, p227)

Find the x -coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

sol:

實習課練習 (EXERCISE 4.6)

2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2 .
5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2 .
6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2 .

THOMAS' CALCULUS (12/E)

4.7 Antiderivatives

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Finding Antiderivatives

1.1 We want to find a function F from its _____ . _____ If such a function F exists, it is called an _____ of f .

1.2 *Definition: Antiderivative*

A function F is an antiderivative of f on an interval I if _____ for all x in I .

1.3 If F is an _____ of f on an interval I , then the most general antiderivative of f on I is _____ where C is an arbitrary constant.

1.4 *Definition: Indefinite Integral, Integrand*

The set of all antiderivatives of f is the _____ of f with respect to x , denoted by _____. The symbol \int _____ is an integral sign. The function f is the _____ of the _____ and x is the variable of _____.

 **Ex. 1** (example1, p230)

Find an antiderivative for each of the following functions.


1. $f(x) = 2x$.

2. $g(x) = \cos x$.

3. $h(x) = 2x + \cos x$.


sol:

NOTE: _____ , _____ ,
 _____ .

 **Ex. 2** (example2, p231)

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

sol:

 **Ex. 3** (exampleTABLE4.2, p232)

| Function | General antiderivative |
|--------------------|-------------------------------------|
| 1. x^n | _____, $n \neq -1, n$ rational. |
| 2. $\sin kx$ | _____, k a constant, $k \neq 0$. |
| 3. $\cos kx$ | _____, k a constant, $k \neq 0$. |
| 4. $\sec^2 x$ | _____. |
| 5. $\csc^2 x$ | _____. |
| 6. $\sec x \tan x$ | _____. |
| 7. $\csc x \cot x$ | _____. |

 **Ex. 4** (example3, p231)

Find an general antiderivative of each of the following functions.

(a) $f(x) = x^5$.

(b) $g(x) = \frac{1}{\sqrt{x}}$.

(c) $h(x) = \sin 2x.$

(d) $i(x) = \cos \frac{x}{2}.$


sol:

2 Indefinite Integrals

2.1 Definition

The collection of all _____ of f is called the _____ of f with respect to x , and is denoted by _____

The symbol \int is an _____. The function f is the _____ of the integral, and x is the variable of integration.

 **Ex. 5** (example6, p235)

Evaluate $\int (x^2 - 2x + 5) dx.$

sol:

實習課練習 (EXERCISE 4.7)

Finding antiderivatives.

6. (a) $-\frac{2}{x^3}$ (b) $\frac{1}{2x^3}$, (c) $x^3 - \frac{1}{x^3}$.

9. (a) $\frac{2}{3}x^{-1/3}$ (b) $\frac{1}{3}x^{-2/3}$, (c) $-\frac{1}{3}x^{-4/3}$.

15. (a) $\csc x \cot x$ (b) $\csc 5x \cot 5x$ (c) $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$

Finding indefinite integrals.

28. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) dx$

32. $\int x^{-3}(x+1) dx$

37. $\int 7 \sin \frac{\theta}{3} d\theta$

42. $\int \frac{2}{5} \sec \theta \tan \theta d\theta$

50. $\int (2 + \tan^2 \theta) d\theta$

53. $\int \cos \theta (\tan \theta + \sec \theta) d\theta$

THOMAS' CALCULUS (12/E)

5.1~ 5.2 Estimating with Finite Sums, Limits of Finite Sums

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Approximating Area

1.1 What is the area of the shaded region R that lies above the x -axis, below the graph of $y = 1 - x^2$ and between the vertical lines $x = 0$ and $x = 1$? (圖示如下:)

(a) **upper sum:**

(b) **lower sum:**

(c) **midpoint rule:**

2 Finite Sums and Sigma Notation

2.1 $a_1 + a_2 + \cdots + a_{n-1} + a_n = \quad = \quad .$

2.2 Examples

(a) $\sum_{k=1}^5 k = \underline{\hspace{2cm}}$

(b) $\sum_{k=1}^3 (-1)^k k = \underline{\hspace{2cm}}$

(c) $\sum_{k=1}^2 \frac{k}{k+1} = \underline{\hspace{2cm}}$

(d) $\sum_{i=4}^5 \frac{i^2}{i-1} = \underline{\hspace{2cm}}$

2.3 Algebra Rules for Finite Sums

(a) Sum Rule: $\sum (a_k + b_k) = \underline{\hspace{2cm}}$

(b) Difference Rule: $\sum (a_k - b_k) = \underline{\hspace{2cm}}$

(c) Constant Multiple Rule: $\sum ca_k = \underline{\hspace{2cm}}$ (Any number c)

(d) Constant Value Rule: $\sum c = \underline{\hspace{2cm}}$

2.4 Examples:

(a) $\sum_{k=1}^n (3k - k^2) =$

(b) $\sum_{k=1}^n (-a_k) =$

(c) $\sum_{k=1}^3 (k + 4) =$

(d) $\sum_{k=1}^n \frac{1}{n} =$

 **Ex. 1** (example4, p258)

Show that $\sum_{k=1}^n k = \frac{n(n+1)}{2}$.

sol:

NOTE:

1. The first n squares: $\sum_{k=1}^n k^2 =$ _____ .

2. The first n cubes: $\sum_{k=1}^n k^3 =$ _____ .

3 Riemann Sums

3.1 Let $n - 1$ points $\{x_1, x_2, \dots, x_{n-1}\}$ between a and b and satisfying

3.2 A **partition** of $[a, b]$: _____ .

3.3 The k th subinterval of P is _____ .

3.4 The _____ of a partition P , _____, the largest of all subinterval widths.

3.5 A _____ for f on the interval $[a, b]$:

for every $c_k \in [x_{k-1}, x_k]$, $k = 1, \dots, n$. (圖示如下)

實習課練習 (EXERCISE 5.1)

3. Use finite approximation to estimate the area under the graph of the function $f(x) = \frac{1}{x}$ using (a) a lower sum; (b) an upper sum; and (c) midpoint rule with four rectangles of equal width.

實習課練習 (EXERCISE 5.2)

21. Evaluate $\sum_{k=1}^7 (-2k)$.

26. Evaluate $\sum_{k=1}^7 k(2k + 1)$.

27. Evaluate $\sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k\right)^3$.

32. Evaluate (a) $\sum_{k=1}^5 \frac{k^3}{225} + \left(\sum_{k=1}^5 k\right)^3$, (b) $\sum_{k=1}^n \frac{c}{n}$, (c) $\sum_{k=1}^n \frac{k}{n^2}$.

34. Graph $f(x) = -x^2$ on $[0, 1]$. Partition the interval into four subintervals of equal length. Add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^4 f(c_k)\Delta x_k$, given that c_k is the midpoint of the k th subinterval.

THOMAS' CALCULUS (12/E)

5.3 The Definite Integral

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Limits of Riemann Sums

1.1 Definitions: The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the _____ of f over $[a, b]$ and that I is the _____ of the _____ if the following condition is satisfied:

Given any number $\delta > 0$ there is a corresponding number such that for every partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with _____ and any choice of c_k in $[x_k, x_{k+1}]$, we have

1.2 The symbol for the number I in the definition of the definite integral is

which is read as " _____ " or sometimes as " _____ "

1.3 When the definition is satisfied, we say the Riemann sums of f on $[a, b]$ _____ to the definite integral $I = \int_a^b f(x) dx$ and that f is _____ over $[a, b]$.

1.4 When each partition has n equal subintervals, each of width _____,

1.5 The _____ : $t, u, x: \int_a^b f(t) dt, \int_a^b f(u) du, \int_a^b f(x) dx.$

1.6 *Theorem 1: The Existence of Definite Integrals*

A _____ function is integrable. That is, if a function f is _____ on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

2 Properties of Definite Integrals

2.1 When f and g are integrable, the definite integral satisfies following.

(a) Order of Integration: $\int_a^b f(x) dx =$ _____

(b) Zero Width Interval: $\int_a^a f(x) dx =$ _____

(c) Constant Multiple: $\int_a^b kf(x) dx =$ _____

(d) Sum and Difference: $\int_a^b (f(x) \pm g(x)) dx =$ _____

(e) Additivity: $\int_a^b f(x) dx + \int_b^c f(x) dx =$ _____

(f) Max-Min Inequality: If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then

(g) Domination:

i. $f(x) \geq g(x)$ on $[a, b] \Rightarrow$ _____ .

ii. $f(x) \geq 0$ on $[a, b] \Rightarrow$ _____ .

3 Area Under the Graph of a Nonnegative Function

3.1 Definitions: Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the _____ over $[a, b]$ is the _____ of f from a to b ,


$$A = \underline{\hspace{2cm}}.$$

3.2 Equations

(a) $\int_a^b x \, dx = \underline{\hspace{2cm}}.$

(b) $\int_a^b c \, dx = \underline{\hspace{2cm}}.$

(c) $\int_a^b x^2 \, dx = \underline{\hspace{2cm}}.$

 **Ex. 1** (example4, p268)

Compute $\int_0^b x \, dx$ and find the area under $y = x$ over the interval $[0, b]$, $b > 0$.

sol:


4 Average Value of a Continuous Function

4.1 Definitions: Average or Mean Value of a Function

If f is integrable on $[a, b]$, then its average value (mean value) on $[a, b]$, is

$$\frac{1}{b-a} \int_a^b f(x) dx$$

4.2 The average value of a nonnegative continuous function f over an interval $[a, b]$ is

 **Ex. 2** (example5, p270)

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

sol:

實習課練習 (EXERCISE 5.3)

□ Express the limits as definite integrals.

1. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n c_k^2 \Delta x_k$, where P is a partition of $[0, 2]$.
4. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \left(\frac{1}{c_k}\right) \Delta x_k$, where P is a partition of $[1, 4]$.
6. $\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \sqrt{4 - c_k^2} \Delta x_k$, where P is a partition of $[0, 1]$.

□ Evaluate the integrals.

29. $\int_1^{\sqrt{2}} x \, dx$
34. $\int_0^{0.3} s^2 \, ds$
44. $\int_0^{\sqrt{2}} (t - \sqrt{2}) \, dt$
45. $\int_2^1 \left(1 + \frac{z}{2}\right) \, dz$
49. $\int_1^0 (3x^2 + x - 5) \, dx$

□ Find the area of the region between the given curve and the x -axis on the interval $[0, b]$

52. $y = \pi x^2$.

54. $y = \frac{x}{2} + 1$.

□ Find the average value over the given interval.

56. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$.

61. $g(x) = |x| - 1$ on (a) $[-1, 1]$, (b) $[1, 3]$, (c) $[-1, 3]$.

THOMAS' CALCULUS (12/E)

5.4 The Fundamental Theorem of Calculus

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Mean Value Theorem for Definite Integrals

1.1 Theorem 3: Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

 **Ex. 1** (example2, p275)

Show that if f is continuous on $[a, b]$, $a \neq b$, and if $\int_a^b f(x) dx = 0$, then $f(x) = 0$ at least once in $[a, b]$.

sol:

2 Fundamental Theorem, Part 1


2.1 Theorem 4: The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then

(a) $F(x) =$ _____ is continuous on $[a, b]$ and differentiable on (a, b) ,

(b) and its derivative is _____ ;

$$F'(x) = f(x).$$

 **Ex. 2** (example2, p276)

Use the Fundamental Theorem to find dy/dx if

$$1. y = \int_a^x (t^3 + 1) dt$$

$$2. y = \int_x^5 3t \sin t dt$$

$$3. y = \int_1^{x^2} \cos t dt$$

3 Fundamental Theorem, Part 2 (The Evaluation Theorem)

3.1 Theorem 4: The Fundamental Theorem of Calculus Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

 **Ex. 3** (example3, p278)

$$1. \int_0^{\pi} \cos x dx =$$

$$2. \int_{-\pi/4}^0 \sec x \tan x \, dx =$$

$$3. \int_1^4 \left(\frac{3}{2} \sqrt{x} - \frac{4}{x^2} \right) dx =$$

4 Total Area

4.1 To compute the _____ bounded by the graph of a function and the x -axis requires more care when the function takes on both _____ and _____ values.

4.2 We must be careful to _____ into subintervals on which the function _____.

4.3 *Summary:*

To find the area between the graph of _____ and the x -axis over the interval $[a, b]$, do the following: (a) Subdivide _____. (b) Integrate _____. (c) Add _____.



 **Ex. 4** (example6, p280)

Figure 5.20 shows the graph of $f(x) = x^2 - 4$ and its mirror image $g(x) = 4 - x^2$ reflected across the x -axis. For each function, compute (a) the definite integral over the interval $[-2, 2]$ and (b) the area between the graph and the x -axis over $[-2, 2]$.

sol:

 **Ex. 5** (example6, p280)

Let $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute (a) the definite integral of $f(x)$ over $[0, 2\pi]$ and (b) the area between the graph $f(x)$ and the x -axis over $[0, 2\pi]$.

sol:

 **Ex. 6** (example8, p281)

Find the area of the region between the x -axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

sol:

實習課練習 (EXERCISE 5.4)

□ Evaluate the integrals.

$$1. \int_{-2}^0 (2x + 5) dx$$

$$8. \int_1^{32} x^{-6/5} dx$$

$$14. \int_{-\pi/3}^{\pi/3} \frac{1 - \cos 2t}{2} dt$$

$$19. \int_1^{-1} (r + 1)^2 dr$$

$$24. \int_1^8 \frac{(x^{1/3} + 1)(2 - x^{2/3})}{x^{1/3}} dx$$

$$27. \int_{-4}^4 |x| dx$$

□ Find the derivatives (a) by evaluating the integral and differentiating the result. (b) by differentiating the integral directly.

$$29. \frac{d}{dx} \int_0^{\sqrt{x}} \cos t dt$$

$$30. \frac{d}{dx} \int_1^{\sin x} 3t^2 dt$$

□ Find dy/dx .

$$33. y = \int_0^x \sqrt{1 + t^2} dt$$

$$39. y = \int_0^{\sin x} \frac{dt}{\sqrt{1 - t^2}}, |x| < \frac{\pi}{2}$$

□ Find the total area between the region and the x -axis.

$$41. y = -x^2 - 2x, -3 \leq x \leq 2$$

$$44. y = x^{1/3} - x, -1 \leq x \leq 8$$

THOMAS' CALCULUS (12/E)

5.5 Indefinite Integrals and the Substitution Rule

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 The Power Rule in Integral Form

1.1 If u is a differentiable function of x and n is a rational number, $x \neq -1$, the Chain Rule tells us that

$$\frac{d}{dx} \left(\frac{u^{n+1}}{n+1} \right) = \underline{\hspace{2cm}}.$$


1.2 If u is any differentiable function, then

$$\int u^n du = \underline{\hspace{2cm}}, \quad (n \neq -1, n \text{ rational})$$

 **Ex. 1** (example1, p285)

Find the integral $\int (x^3 + x)^5 (3x^2 + 1) dx$

sol:

 **Ex. 2** (example2, p285)

Find the integral $\int \sqrt{2x+1} dx$

sol:

2 Substitution: Running the Chain Rule Backwards

2.1 Theorem 6: The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

- (a) Substitute _____ and _____ to obtain the integral _____.
- (b) Integrate with respect to _____.
- (c) Replace _____ by _____ in the result.

 **Ex. 3** (example3, p287)


Find $\int \sec^2(5t + 1) \cdot 5 dt$

sol:

 **Ex. 4** (example4, p287)


Find $\int \cos(7\theta + 3) d\theta$

sol:

 **Ex. 5** (example5, p287)


Evaluate $\int x^2 \sin(x^3) dx$

sol:

 Ex. 6 (example6, p288)

Evaluate $\int x\sqrt{2x+1} dx$.

sol:

 Ex. 7 (example7, p288)

Evaluate $\int \frac{2z}{\sqrt[3]{z^2+1}} dz$

sol:

3 The Integrals of $\sin^2 x$ and $\cos^2 x$

 Ex. 8 (example7, p7)373

(a) $\int \sin^2 x dx$

(b) $\int \cos^2 x dx$

實習課練習 (EXERCISE 5.5)

8.
$$\int x \sin(2x^2) dx$$

11.
$$\int \frac{9r^2}{\sqrt{1-r^3}} dr$$

13.
$$\int \sqrt{x} \sin^2(x^{3/2} - 1) dx$$

16.
$$\int \frac{dx}{\sqrt{5x+8}}$$

19.
$$\int \theta \sqrt[4]{1-\theta^2} d\theta$$

21.
$$\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$$

25.
$$\int \sin^5 \frac{x}{3} \cos \frac{x}{3} dx$$

31.
$$\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt$$

38.
$$\int \sqrt{\frac{x-1}{x^5}} dx$$

46.
$$\int (x+5)(x-5)^{1/3} ds$$

48.
$$\int 3x^5 \sqrt{x^3+1} dx$$

50.
$$\int \frac{x}{(x-4)^3} dx$$

THOMAS' CALCULUS (12/E)

5.6 Substitution and Area Between Curves

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Substitution Formula1.1 *Theorem 7: Substitution in Definite Integrals*


If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g , then

 **Ex. 1** (example1, p292)

Evaluate $\int_{-1}^1 3x^2 \sqrt{x^3 + 1} dx$.

sol(1):

sol(2):

 **Ex. 2** (example2, p293)

Evaluate $\int_{\pi/4}^{\pi/2} \cot \theta \csc^2 \theta \, d\theta$.

sol:

2 Definite Integrals of Symmetric Functions


- 2.1 (a) f is an even function if _____ .
 (b) f is an odd function if _____ .

2.2 Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^a f(x) \, dx =$ _____ .

(b) If f is odd, then $\int_{-a}^a f(x) \, dx =$ _____ .

 **Ex. 3** (example3, p294)

Evaluate $\int_{-2}^2 (x^4 - 4x^2 + 6) \, dx$.


sol:

3 Areas Between Curves

3.1 Definition: Area Between Curves


If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of _____ from a to b :

$$A = \int_a^b \text{_____} dx$$

 **Ex. 4** (example4, p295)

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

sol:

 **Ex. 5** (example5, p295)


Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x -axis and the line $y = x - 2$.

sol:

4 Integration with Respect to y

4.1 If a region's bounding curves are described by functions of y , the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x . (圖示如下)

4.2 Use the formula $A = \int_{c_1}^{c_2} (f(y) - g(y)) dy$, where f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

 **Ex. 6** (example6, p296)

Find the area of the region in Example 5 by integrating with respect to y .

sol:

實習課練習 (EXERCISE 5.6)

□ Evaluate the integrals.

2. $\int_{-1}^1 r\sqrt{1-r^2} dr$

4. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x dx$

8. $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2} dv$

15. $\int_0^1 \sqrt{t^5+2t}(5t^4+2) dt$

24. $\int_{-1}^{-1/2} t^{-2} \sin^2\left(1+\frac{1}{t}\right) dt$

□ Find the areas of the regions enclosed by the lines and curves.

48. $y = x\sqrt{a^2-x^2}$, $a > 0$ and $y = 0$.

50. $y = |x^2 - 4|$, and $y = (x^2/2) + 4$.

54. $x - y^2 = 0$, and $x + 2y^2 = 3$.

61. $x + 4y^2 = 4$, and $x + y^4 = 1$, for $x \geq 0$.

63. $y = 2\sin x$, and $y = \sin 2x$, $0 \leq x \leq \pi$.

73. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x -axis.

76. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to (a) x , (b) y .

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6.1 Volumes Using Cross-Sections

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Volumes of Solids

- 1.1 A cross-section of a solid S is the _____ formed by _____ S with a _____ (Figure 6. 1).

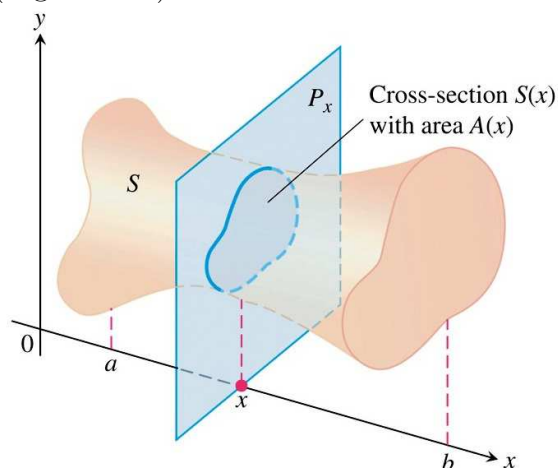


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x -axis through the point x in the interval $[a, b]$.

- 1.2 Three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid: the method of _____, the _____ method, and the _____ method.
- 1.3 (Figure 6.2) If the cylindrical solid has a known _____ and _____, then the volume of the cylindrical solid is

$$\text{Volume} = \underline{\hspace{2cm}}.$$

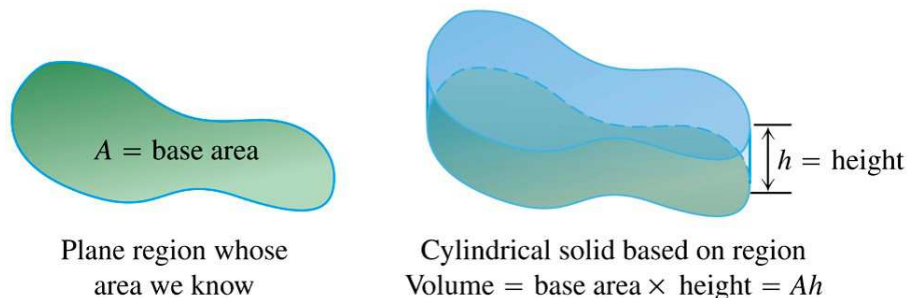


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

1.4 If the cross-section of the solid S at each point x in the interval _____ is a _____ of _____, and A is a continuous function of x , we can define and calculate the volume of the solid S as the _____.

2 Slicing by Parallel Planes

2.1 Partition _____ into subintervals of width (length) _____ and slice the solid by planes _____ to the x -axis at the partition points _____. The planes _____, perpendicular to the x -axis at the partition points, slice S into thin _____ (Figure 6.3).

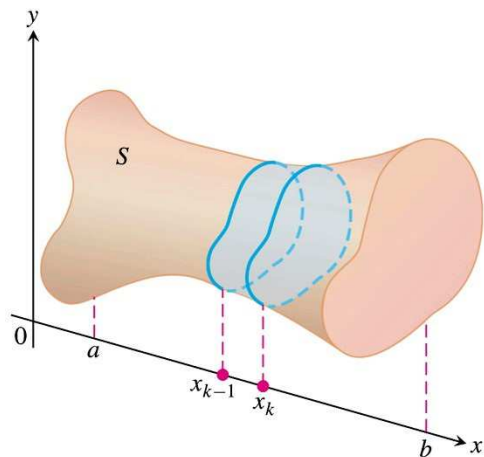
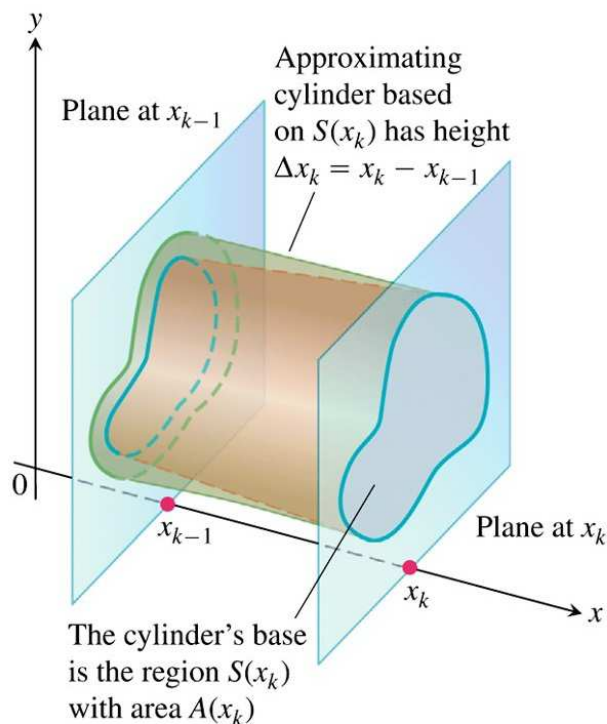


FIGURE 6.3 A typical thin slab in the solid S .

2.2 Approximate the slab between the plane at _____ and the plane at _____ by a _____ with base area _____ and height _____.

(Figure 6.4).



NOT TO SCALE

FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

2.3 The volume V_k of this cylindrical solid is _____, which is approximately the same volume as that of the slab:

Volume of the k th slab _____.

2.4 The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$.

2.5 Taking a partition of $[a, b]$ into _____ subintervals with _____ gives

$$\underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$


2.6 **Definition: The Volume of a Solid**

The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

$$V = \int_a^b A(x) \, dx$$

2.7 **Calculating the Volume of a Solid**

- (a) Sketch the solid and a typical _____.
- (b) Find a formula for _____, the _____ of a typical cross-section.
- (c) Find the _____ of integration.
- (d) Integrate _____ to find the volume.

 **Ex. 1** (example1, p309)

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

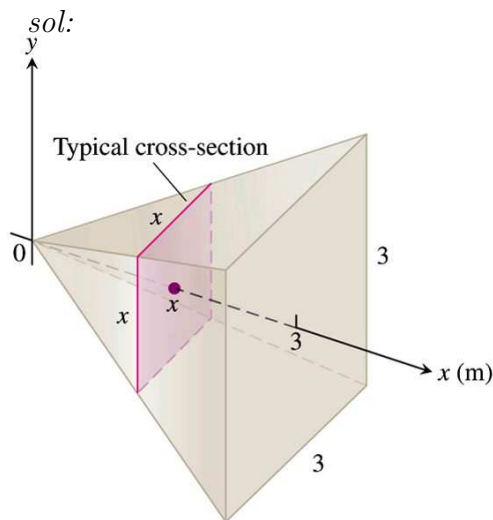


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

 **Ex. 2** (example2, p310)

A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

sol:

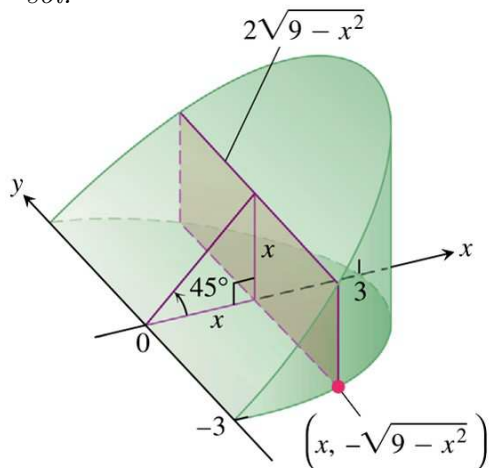


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x -axis. The cross-sections are rectangles.

3 Solids of Revolution: The Disk Method

3.1 The solid generated by _____ (or _____) a plane region about an _____ in its plane is called _____.

3.2 To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area _____ is the area of _____, the

distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

3.3 Volume by Disks for Rotation About the x -axis

$$V = \underline{\hspace{2cm}} = \underline{\hspace{2cm}} .$$

3.4 To find the volume of a solid generated by revolving a region between the and a curve , , about the y -axis, we use the same method with x replaced by y , the circular cross-section is

$$A(y) = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$$

3.5 Volume by Disks for Rotation About the y -axis

$$V = \underline{\hspace{2cm}} = \underline{\hspace{2cm}} .$$

 **Ex. 3** (example4, p311)

The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x -axis is revolved about the x -axis to generate a solid. Find its volume.

sol:

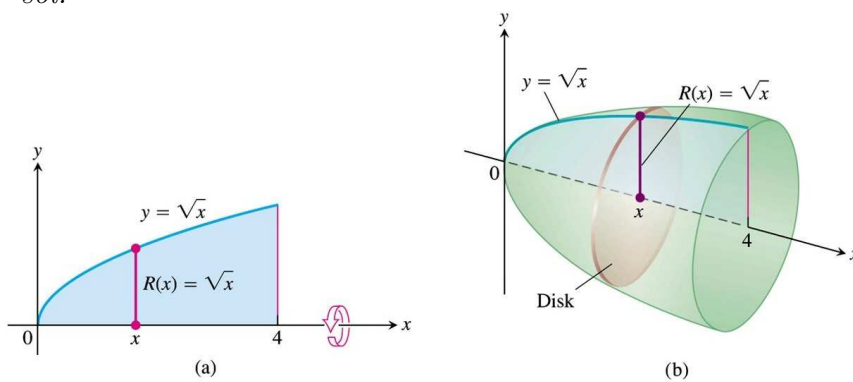



FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

 **Ex. 4** (example6, p311)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

sol:

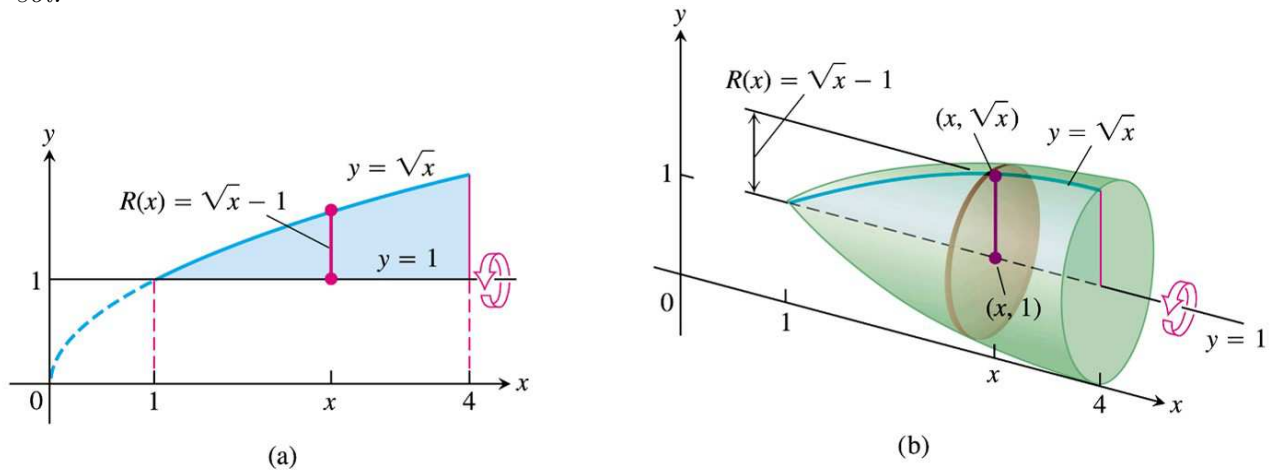



FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6.

 **Ex. 5** (example7, p313)

Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y -axis.

sol:

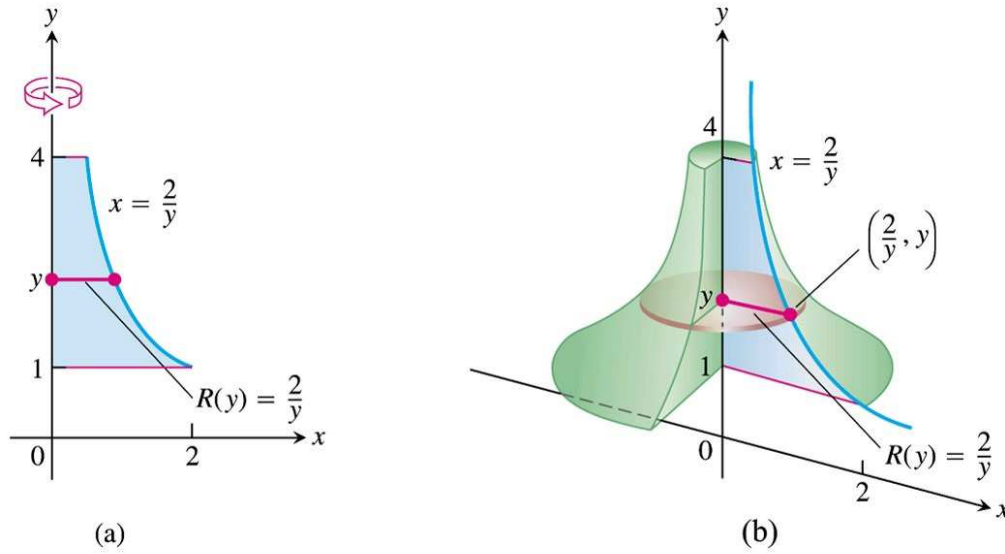



FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

 **Ex. 6** (example8, p313)

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

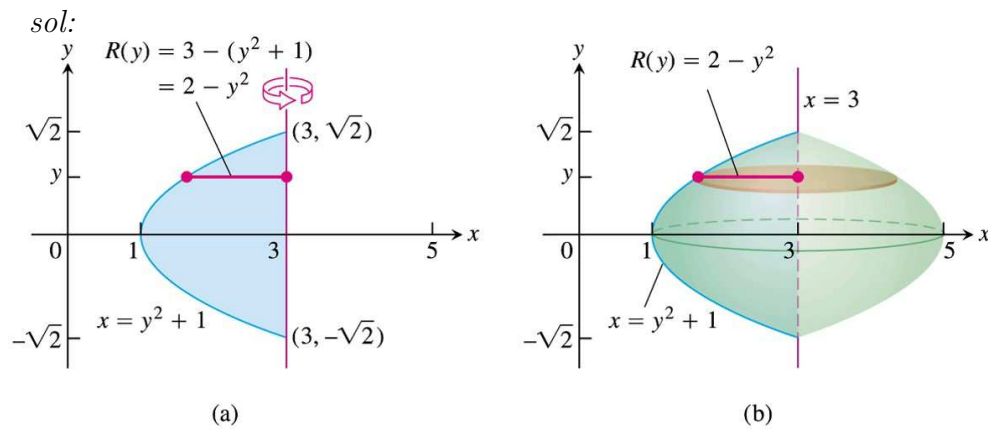


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

4 Solids of Revolution: The Washer Method

4.1 If the region we revolve to generate a solid does not _____ or _____, the solid has a _____ in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are _____ instead of disks. The dimensions of a typical washer are: _____, _____.

4.2 The washer's area is

$$A(x) = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$

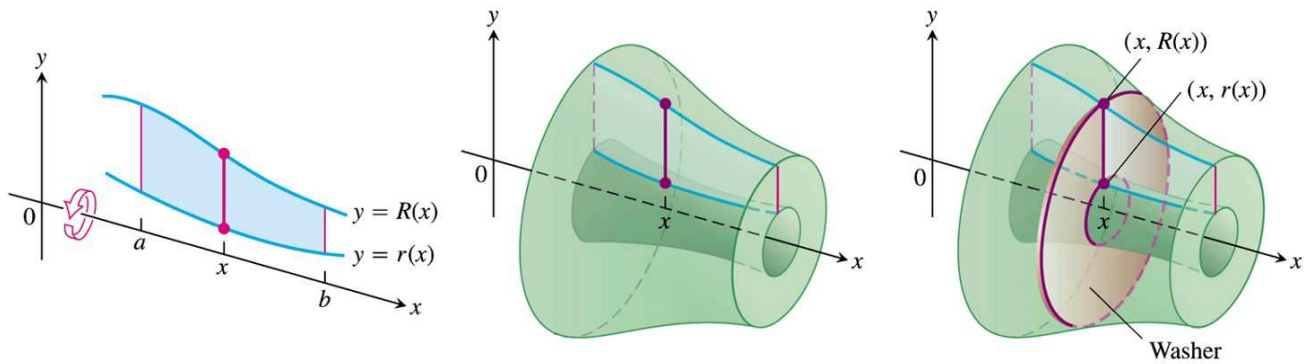



FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x) dx$ leads to a slightly different formula.

4.3 Volume by Washers for Rotation About the x -axis

$$V = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$

 **Ex. 7** (example9, p314)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

sol:

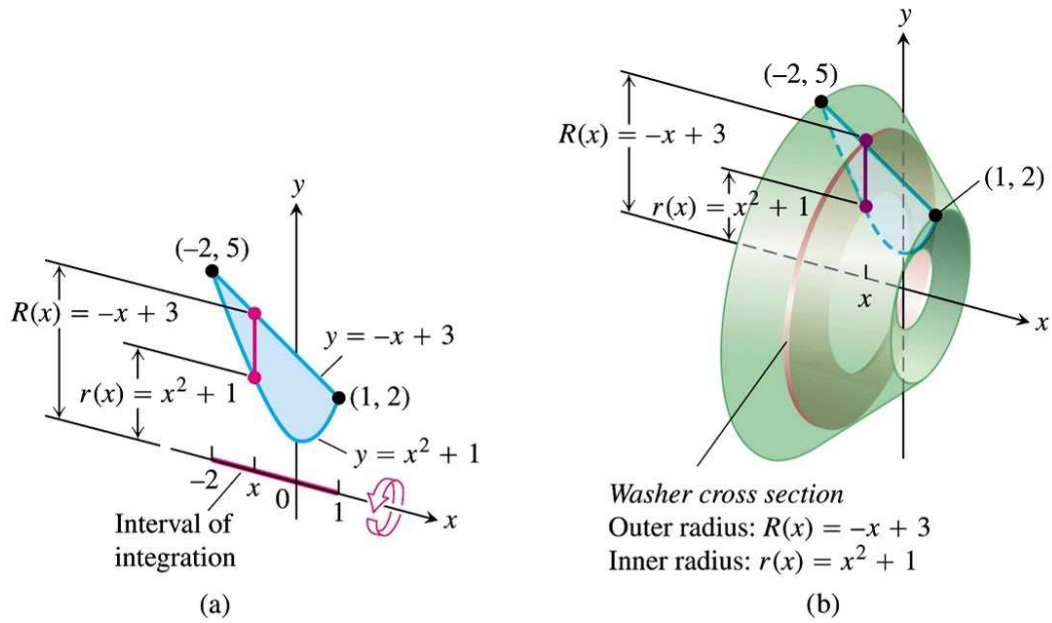



FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x -axis, the line segment generates a washer.

 **Ex. 8** (example10, p315)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.

sol:

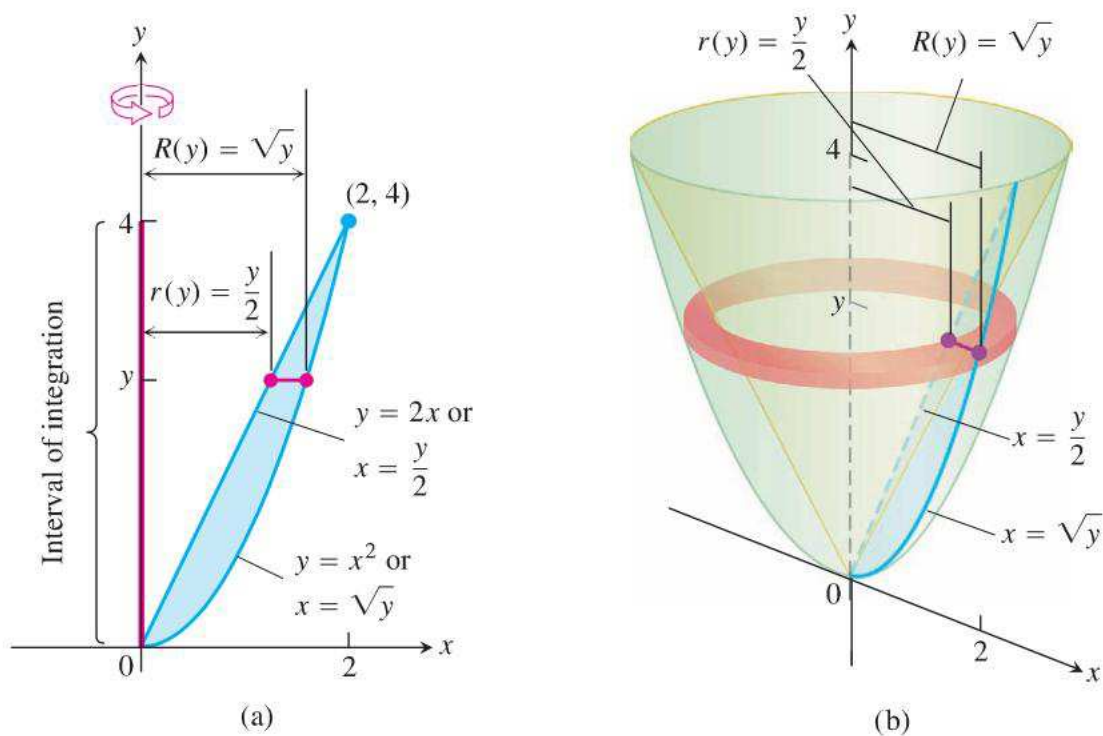
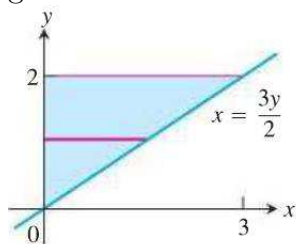


FIGURE 6.15 (a) The region being rotated about the y -axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

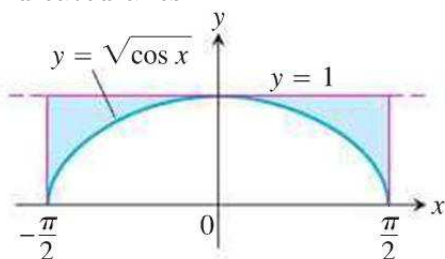
實習課練習 (EXERCISE 6.1)

In Exercises 1-10, Find the volumes of the solids.

3. The solid lies between planes perpendicular to the x -axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x -axis between these planes are squares whose bases run from the semi-circle $y = -\sqrt{1-x^2}$ to the semicircle $y = \sqrt{1-x^2}$.
9. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x/2$. The cross-sections perpendicular to the x -axis are
 - a. isosceles triangles of height 6.
 - b. semi-circles with diameters running across the base of the solid.
16. Find the volume of the solid generated by revolving the shaded region about the given axis. About the y -axis.



22. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x -axis. $y = x - x^2, y = 0$.
26. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x, 0 \leq x \leq \pi/2$, and on the left by the y -axis, about the line $y = 2$.
27. Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the y -axis. $x = \sqrt{2y}/(y^2 + 1), x = 0, y = 1$.
33. Find the volumes of the solids generated by revolving the shaded regions about the indicated axes.



47. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about
- a. the x -axis. b. the y -axis. c. the line $y = 2$. d. the line $x = 4$.
50. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about
- a. the x -axis. b. the y -axis.

THOMAS' CALCULUS (12/E)

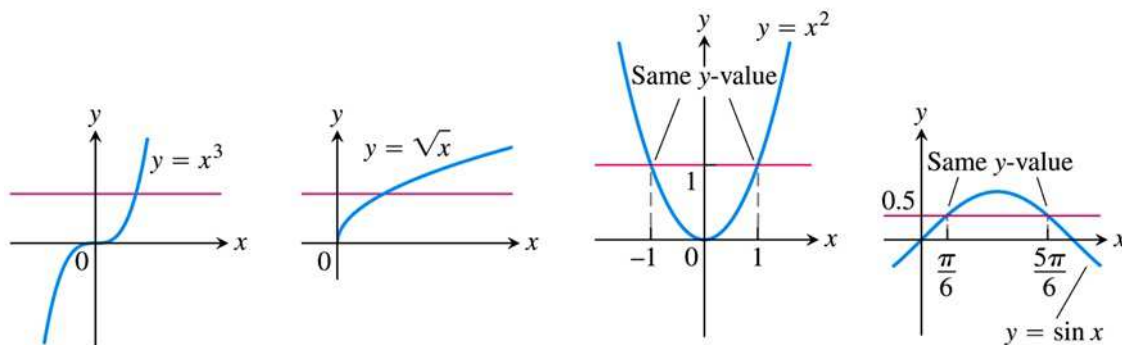
7.1 Inverse Function and Their Derivatives

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>**1 One-to-One Functions and Inverse Functions**1.1 *Definitions: One-to-One Function*

A function $f(x)$ is one-to-one on a domain D if _____ whenever _____ in D .

1.2 One-to-one: (a) $y = x^3$, (b) $y = \sqrt{x}$. (圖示如下)Not one-to-one: (c) $y = x^2$, (d) $y = \sin x$. (圖示如下)

1.3 A function $y = f(x)$ is one-to-one if and only if its graph intersects each _____ at most _____.

1.4 *Definitions: Inverse Function*

Suppose that f is a one-to-one function on a domain D with range R . The inverse function _____ is defined by

if _____

The _____ of f^{-1} is _____ and the _____ of f^{-1} is _____.

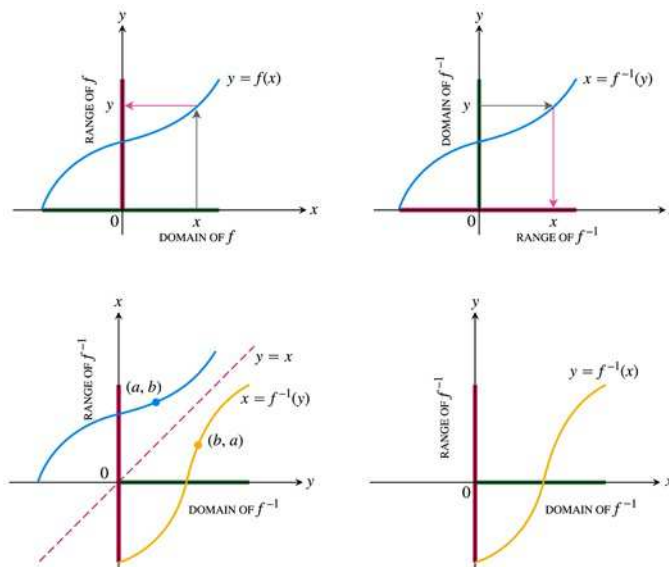
1.5 $(f^{-1} \cdot f)(x) = \underline{\hspace{2cm}}$ for all x in the domain of f .

1.6 $(f \cdot f^{-1})(y) = \underline{\hspace{2cm}}$ for all y in the domain of f^{-1} .

1.7 Only a one-to-one function can have an .

2 Finding Inverses

2.1 Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$. (圖示如下)



2.2 Pass from f to f^{-1} .

- (a) Solve the equation for x . This gives a formula where x is expressed as a function of y .
- (b) Interchange , obtaining a formula where f^{-1} is expressed in the conventional format with x as the variable and y as the .

 **Ex. 1** (example3, p364)

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x .

sol:

 **Ex. 2** (example4, p364)

Find the inverse of the function $y = x^2$, $x \geq 0$, expressed as a function of x .

sol:

3 Derivatives of Inverses of Differentiable Functions

3.1 $f(x) = (1/2)x + 1$ and $f^{-1}(x) = \underline{\hspace{2cm}}$.

$$\frac{d}{dx}f(x) = \underline{\hspace{2cm}}$$

$$\frac{d}{dx}f^{-1}(x) = \underline{\hspace{2cm}}$$

3.2 *Theorem 1: The Derivative Rule for Inverses*


(a) If f has an interval I as domain and $f'(x)$ exists and is never zero on I , then f^{-1} is $\underline{\hspace{2cm}}$ at every point in its domain.

(b) The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the $\underline{\hspace{2cm}}$ of f' the value of at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \underline{\hspace{2cm}} \quad \text{or} \quad \left. \frac{d}{dx}f^{-1} \right|_{x=b} = \underline{\hspace{2cm}}$$


3.3 When $y = f(x)$ is differentiable at $x = a$ and we change x by a small amount dx , the corresponding change in y is approximately $\underline{\hspace{2cm}}$. This means that y changes about $\underline{\hspace{2cm}}$ times as fast as x when $x = a$ and that x changes about $\underline{\hspace{2cm}}$ times as fast as y when $y = b$.

3.4 It is reasonable that the derivative of f^{-1} at b is the $\underline{\hspace{2cm}}$ of the derivative of f at a .

 **Ex. 3** (example5, p366)

Apply The Derivative Rule for Inverse Theorem to the function $f(x) = x^2, x \geq 0$.

sol:

 **Ex. 4** (example6, p366)

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

sol:

實習課練習 (EXERCISE 7.1)

21. Let $f(x) = x^3 - 1$. Find a formula for f^{-1} .
22. Let $f(x) = x^2 - 2x + 1$, $x \geq 1$. Find a formula for f^{-1} .
33. Let $f(x) = x^2 - 2x$, $x \leq 1$. Find f^{-1} and identify the domain and range of f^{-1} .
37. Let $f(x) = 5 - 4x$, $a = 1/2$. Find $f^{-1}(x)$. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$.
38. Let $f(x) = 2x^2$, $x \geq 0$, $a = 5$. Find $f^{-1}(x)$. Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$.
41. Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

THOMAS' CALCULUS (12/E)

7.2 Natural Logarithms

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Definition of the Natural Logarithm Function

1.1 The natural logarithm of a positive number x , written as _____, is the value of an integral.

1.2 *Definitions: The Natural Logarithm Function*

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$

1.3 If $x > 1$, then $\ln x$ is the area under the curve _____ from _____ to _____.

1.4 For $0 < x < 1$, $\ln x$ gives the _____ under the curve from _____ to _____.

1.5 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$.

The graph of the logarithm rises above the x -axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left. (圖示如下)

1.6 *Definitions: The Number e*

The number e is that number in the domain of the natural logarithm satisfying _____.

1.7 $\ln 1 =$ _____.

1.8 Geometrically, the number e corresponds to the point on the x -axis for which the area under the graph of _____ and above the interval _____ is the exact area of the unit square.

2 The Derivative of $y = \ln x$

2.1 $\frac{d}{dx} \ln x =$ _____ $=$ _____

2.2 $y = \ln u$
 $\frac{d}{dx} \ln u =$ _____ $=$ _____, $u > 0$

 **Ex. 1** (example1, p371)

1. $\frac{d}{dx} \ln 2x =$
2. $\frac{d}{dx} \ln(x^2 + 3) =$
3. $\frac{d}{dx} \ln ax =$

3 Properties of Logarithms

3.1 *Theorem 2: Properties of Logarithms*

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

- (a) Product Rule: _____
- (b) Quotient Rule: _____
- (c) Reciprocal Rule: _____
- (d) Power Rule: _____

 **Ex. 2** (example2, p372)

1. $\ln 4 + \ln \sin x =$

2. $\ln \frac{x+1}{2x-3} =$

3. $\ln \sec x =$

4. $\ln \sqrt[3]{x+1} =$

4 The Integral $\int(1/u)du$

4.1 If u is a differentiable function that is never zero,

$$\int \frac{1}{u} du = \underline{\hspace{2cm}}.$$

4.2 If $u = f(x)$, then $du = \underline{\hspace{2cm}}$ and

$$\int \frac{1}{u} du = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}.$$

 **Ex. 3** (example3, p374)

$$\int_0^2 \frac{2x}{x^2-5} dx$$

sol:

5 The Integral of $\tan x$, $\cot x$, $\sec x$ and $\csc x$

5.1 $\int \tan u du = \underline{\hspace{2cm}}$

Proof:

$$5.2 \int \cot u \, du = \underline{\hspace{2cm}}$$


Proof:

$$5.3 \int \sec u \, du = \underline{\hspace{2cm}}$$

Proof:

$$5.4 \int \csc u \, du = \underline{\hspace{2cm}}$$


Proof:

 **Ex. 4** (example4, p375)

$$\int_0^{\pi/6} \tan 2x \, dx$$

sol:

6 Logarithmic Differentiation

 Ex. 5 (example5, p375)

Find dy/dx if $y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}$, $x > 1$.

sol:

實習課練習 (EXERCISE 7.2)

2. Express the following logarithms in terms of $\ln 5$ and $\ln 7$. (a) $\ln(1/125)$, (b) $\ln 9.8$,
(c) $\ln 7\sqrt{7}$, (d) $\ln 1225$, (e) $\ln 0.056$, (f) $(\ln 35 + \ln(1/7))/(\ln 25)$.

3. Simplify the expressions: (a) $\ln \sin \theta - \ln\left(\frac{\sin \theta}{5}\right)$, (b) $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$,
(c) $\frac{1}{2} \ln(4t^4) - \ln 2$.

Find the derivative of y with respect to x , or t , as appropriate.

8. $y = \ln(t^3/2)$

15. $y = t(\ln t)^2$

22. $y = \frac{x \ln x}{1 + \ln x}$

35. $y = \int_{x^2/2}^{x^2} \ln \sqrt{t} dt$

Evaluate the integrals:

38. $\int_{-1}^0 \frac{3}{3x-2} dx$

44. $\int_2^4 \frac{dx}{x \ln x}$

45. $\int_2^4 \frac{dx}{x(\ln x)^2}$

52. $\int_0^{\pi/12} 6 \tan 3x dx$

53. $\int \frac{dx}{2\sqrt{x} + 2x}$

Use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

55. $y = \sqrt{x(x+1)}$

62. $y = \frac{1}{t(t+1)(t+2)}$

67. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$

THOMAS' CALCULUS (12/E)

7.3 Exponential Function

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 The Inverse of $\ln x$ and e^x

1.1 The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. (圖示如下)

1.2 $\ln(e) = \underline{\hspace{2cm}}$, $e = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$.

1.3 $e \doteq \underline{\hspace{2cm}}$.

1.4 *Definitions: The Natural Exponential Function*


For every real number x , $e^x = \underline{\hspace{2cm}} = \underline{\hspace{2cm}}$.

1.5 $\ln e^r = \underline{\hspace{2cm}} \Rightarrow e^r = \underline{\hspace{2cm}}$.

1.6 Inverse Equation for e^x and $\ln x$

$$e^{\ln x} = \underline{\hspace{2cm}}, \quad \forall x > 0$$

$$\ln(e^x) = \underline{\hspace{2cm}}, \quad \forall x$$

 **Ex. 1** (example1, p378)

Solve the equation $e^{2x-6} = 4$ for x .

sol:

2 The Derivative and Integral of e^x

2.1 Let $f(x) = \ln x$ and $y = e^x = \underline{\hspace{2cm}}$. Then $\frac{dy}{dx} = \underline{\hspace{2cm}}$

2.2 If u is any differentiable function of x , then $\frac{d}{dx} e^u = \underline{\hspace{2cm}}$

2.3 $\int e^u du = \underline{\hspace{2cm}}$


 **Ex. 2** (example2, p379)

(a) $\frac{d}{dx} (5e^x) =$

(b) $\frac{d}{dx} e^{-x} =$

(c) $\frac{d}{dx} e^{\sin x} =$

(d) $\frac{d}{dx} (e^{\sqrt{3x+1}}) =$

 **Ex. 3** (example3, p379)

(a) $\int_0^{\ln 2} e^{3x} dx =$

(b) $\int_0^{\pi/2} e^{\sin x} \cos x dx =$

3 Laws of Exponents

3.1 Definitions: General Exponential Functions

For any numbers $a > 0$ and x , the exponential function with base a is _____.

3.2 Theorem 3: Laws of Exponents for e^x

For all numbers x, x_1 and x_2 , the natural exponential e^x obeys the following laws:


(a) $e^{x_1} \cdot e^{x_2} =$ _____

(b) $e^{-1} =$ _____

(c) $e^{x_1}/e^{x_2} =$ _____

(d) $((e^{x_1})^{x_2}) =$ _____

Proof of Law (a):

 **Ex. 4** (example4, p382) Differentiate $f(x) = x^x, x > 0$.

sol:

4 The Number e Expressed as a Limit

4.1 Theorem 4: The Number e as a Limit

The number e can be calculated as the limit: $e = \frac{1}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n}$.

Proof:

4.2 Power Rule (General Form)

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and $\frac{d}{dx}u^n = n u^{n-1} \frac{du}{dx}$.

4.3 Examples:

(a) $\frac{d}{dx}x^{\sqrt{2}} =$

(b) $(2 + \sin 3x)^\pi =$

5 The Derivative of a^u


5.1 If $a > 0$, then $\frac{d}{dx}a^x =$ _____.

Proof:

5.2 If $a > 0$ and u is a differentiable function of x , then a^u is a differentiable function of x and $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$.

5.3 If $a \neq 1$, $\int a^u du =$ _____ .

Proof:

 **Ex. 5** (example5, p383)

(a) $\frac{d}{dx} 3^x =$

(b) $\frac{d}{dx} 3^{-x} =$

(c) $\frac{d}{dx} 3^{\sin x} =$

(d) $\int 2^x dx =$

(e) $\int 2^{\sin x} \cos x dx =$

6 Logarithms with Base a

6.1 Definitions: $\log_a x$

For any positive number $a \neq 1$, _____ is the inverse function of a^x .

6.2 Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = \underline{\hspace{2cm}}, \quad x > 0$$


$$\log_a(a^x) = \underline{\hspace{2cm}}, \forall x$$

6.3 $\log_a x =$ _____

Proof:

6.4 Derivatives and Integrals Involving $\log_a x$:

$$\frac{d}{dx}(\log_a u) = \underline{\hspace{2cm}}$$

 **Ex. 6** (example6, p385)

(a) $\frac{d}{dx} \log_{10}(3x + 1) =$

(b) $\int \frac{\log_2 x}{x} dx =$

實習課練習 (EXERCISE 7.3)

□ Solve for t .

2. (a) $e^{-0.01t} = 1000$, (b) $e^{kt} = \frac{1}{10}$, (c) $e^{(\ln 2)x} = 1/2$.

4. $e^{x^2} e^{2x+1} = e^t$

□ Find the derivative of y with respect to x , t or θ , as appropriate.

9. $y = xe^x - e^x$

14. $y = \ln(3\theta e^{-\theta})$

20. $y = \ln\left(\frac{\sqrt{\theta}}{1 + \sqrt{\theta}}\right)$

24. $y = \int_{e^4\sqrt{x}}^{e^{2x}} \ln t \, dt$

25. $\ln y = e^y \sin x$

28. $\tan y = e^x + \ln x$

□ Evaluate the integrals.

33. $\int 8e^{x+1} \, dx$

41. $\int \frac{e^{1/x}}{x^2} \, dx$

43. $\int_0^{\pi/4} (1 + e^{\tan \theta}) \sec^2 \theta \, d\theta$

49. $\int \frac{e^r}{1 + e^r} \, dr$

$$50. \int \frac{dx}{1+e^x}$$

Find the derivative of y with respect to the given independent variable.

$$57. y = 5^{\sqrt{s}}$$

$$72. y = \log_3 r \cdot \log_9 r$$

$$81. y = \log_2(8t^{\ln 2})$$

Evaluate the integrals.

$$85. \int_0^1 2^{-\theta} d\theta$$

$$91. \int_2^4 x^{2x}(1+\ln x) dx$$

$$96. \int_1^e x^{(\ln 2 - 1)} dx$$

$$99. \int_1^4 \frac{\ln 2 \log_2 x}{x} dx$$

$$106. \int \frac{dx}{x(\log_8 x)^2}$$

$$107. \int_1^{\ln x} \frac{1}{t} dt, \quad x > 1$$

Find the derivative of y with respect to the given independent variable.

$$111. y = x + 1^x$$

$$116. y = x^{\sin x}$$

$$118. y = (\ln x)^{\ln x}$$

THOMAS' CALCULUS (12/E)

7.5 Indeterminate Forms and L'Hopital's Rule

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Indeterminate Form 0/0

1.1 If the continuous functions $f(x)$ and $g(x)$ are both _____ at $x = a$, then _____ cannot be found by substituting $x = a$.

1.2 The substitution produces _____, a meaningless expression, which we cannot evaluate. We use _____ as a notation for an expression known as an _____.

1.3 *Theorem: L'Hopital's Rule (First Form)*

Suppose that _____, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$.

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof:

1.4 *Theorem: L'Hopital's Rule (Stronger Form)*

Suppose that _____, that f and g are differentiable on an open interval I containing a , and that $g'(x) \neq 0$ on I if $x \neq a$. Then


$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

1.5 Using L'Hopital's Rule

To find $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ by L'Hopital's Rule,

- (a) continue to differentiate f and g , so long as we still get the form _____ at $x = a$.
- (b) But as soon as one or the other of these derivatives is different from _____ at $x = a$ we stop differentiating.
- (c) L'Hopital's Rule does not apply when either the _____ or _____ has a finite _____ limit.

 **Ex. 1** (example1, p397)

(a) $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x} =$

(b) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} =$

(c) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} =$

(d) $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} =$

 **Ex. 2** (example2, p398)

Find $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2}$.

sol:

 **Ex. 3** (example3, p398)

(a) $\lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} =$

(b) $\lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} =$

2 Indeterminate Form $\infty/\infty, \infty \cdot 0, \infty - \infty$

2.1 L'Hopital's Rule applies to the indeterminate form _____.


2.2 If _____ and _____ as $x \rightarrow a$ then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \underline{\hspace{2cm}}$$

provided the limit on the right exists.

2.3 In the notation $x \rightarrow a$ may be either _____ or _____.


2.4 Moreover $x \rightarrow a$ may be replaced by the one-sided limits _____ or _____.

 **Ex. 4** (example4, p398)

(a) $\lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x} =$

(b) $\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} =$


$$(c) \lim_{x \rightarrow \infty} \frac{e^x}{x^2} =$$

 **Ex. 5** (example5, p399)

(a) Find $\lim_{x \rightarrow \infty} (x \sin \frac{1}{x})$.

(b) Find $\lim_{x \rightarrow 0^+} (\sqrt{x} \ln x)$.

sol:

 **Ex. 6** (example6, p399)

Find $\lim_{x \rightarrow 0} (\frac{1}{\sin x} - \frac{1}{x})$.

sol:


3 Indeterminate Powers

3.1 If $\lim_{x \rightarrow a} \ln f(x) = L$, then $\lim_{x \rightarrow a} f(x) =$ _____ . Here a may be either finite or infinite.

3.2 Theorem: Cauchy's Mean Value Theorem


Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b) . Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

 Ex. 7 (example7, p400)

Apply l'Hôpital's Rule to show that $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$

sol:

 Ex. 8 (example8, p400)

Find $\lim_{x \rightarrow \infty} x^{1/x}$.

sol:

實習課練習 (EXERCISE 7.5)

Use L'Hôpital Rule to find the limits.

$$5. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}.$$

$$8. \lim_{x \rightarrow -5} \frac{x^2 - 25}{x + 5}.$$

$$14. \lim_{t \rightarrow 0} \frac{\sin 5t}{2t}.$$

$$20. \lim_{x \rightarrow 1} \frac{x - 1}{\ln x - \sin \pi x}.$$

$$27. \lim_{\theta \rightarrow 0} \frac{3^{\sin \theta} - 1}{\theta}.$$

$$29. \lim_{x \rightarrow 0} \frac{x2^x}{2^x - 1}.$$

$$34. \lim_{x \rightarrow 0^+} \frac{\ln(e^x - 1)}{\ln x}.$$

$$41. \lim_{x \rightarrow 1^+} \left(\frac{1}{x - 1} - \frac{1}{\ln x} \right).$$

$$46. \lim_{x \rightarrow \infty} x^2 e^{-x}.$$

$$48. \lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{x \sin x}.$$

$$53. \lim_{x \rightarrow \infty} (\ln x)^{1/x}.$$

$$58. \lim_{x \rightarrow 0} (e^x + x)^{1/x}.$$

$$59. \lim_{x \rightarrow 0^+} x^x.$$

$$60. \lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x} \right)^x.$$

$$62. \lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 2} \right)^{1/x}.$$

$$73. \lim_{x \rightarrow \infty} \frac{e^{x^2}}{xe^x}.$$

THOMAS' CALCULUS (12/E)

7.6 Inverse Trigonometric Functions

開課班級: 通訊 1/電機 1/智財學程微積分

授課教師: 吳漢銘 (國立臺北大學統計學系副教授)

教學網站: <http://www.hmwu.idv.tw>

1 Defining the Inverses

1.1 Domain restrictions that make the trigonometric functions one-to-one.

(a) $y = \sin^{-1} x$ is the number in _____ for which _____.

(b) $y = \cos^{-1} x$ is the number in _____ for which _____.

| $\sin x$ | $\sin^{-1} x$ or $\arcsin x$ |
|----------|------------------------------|
| D: _____ | D: _____ |
| R: _____ | R: _____ |

| $\cos x$ | $\cos^{-1} x$ or $\arccos x$ |
|----------|------------------------------|
| D: _____ | D: _____ |
| R: _____ | R: _____ |

| $\tan x$ | $\tan^{-1} x$ or $\arctan x$ |
|----------------------|------------------------------|
| D: _____ R: _____ | D: _____ R: _____ |

| $\cot x$ | $\cot^{-1} x$ or $\operatorname{arccot} x$ |
|----------------------|--|
| D: _____ R: _____ | D: _____ R: _____ |

| $\sec x$ | $\sec^{-1} x$ or $\operatorname{arcsec} x$ |
|----------------------|--|
| D: _____ R: _____ | D: _____ R: _____ |

| $\csc x$ | $\csc^{-1} x$ or $\operatorname{arccsc} x$ |
|----------------------|--|
| D: _____ R: _____ | D: _____ R: _____ |

1.2 Common values of $\sin^{-1} x$:

| | | | | | | |
|---------------|--------------|--------------|-------|--------|---------------|---------------|
| x | $\sqrt{3}/2$ | $\sqrt{2}/2$ | $1/2$ | $-1/2$ | $-\sqrt{2}/2$ | $-\sqrt{3}/2$ |
| $\sin^{-1} x$ | | | | | | |
| $\cos^{-1} x$ | | | | | | |

TABLE 1.3 Values of $\sin \theta$, $\cos \theta$, and $\tan \theta$ for selected values of θ

| Degrees | -180 | -135 | -90 | -45 | 0 | 30 | 45 | 60 | 90 | 120 | 135 | 150 | 180 | 270 | 360 |
|--------------------|--------|-----------------------|------------------|-----------------------|---|----------------------|----------------------|----------------------|-----------------|----------------------|-----------------------|-----------------------|-------|------------------|--------|
| θ (radians) | $-\pi$ | $-\frac{3\pi}{4}$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{4}$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | π | $\frac{3\pi}{2}$ | 2π |
| $\sin \theta$ | 0 | $-\frac{\sqrt{2}}{2}$ | -1 | $-\frac{\sqrt{2}}{2}$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | -1 | 0 |
| $\cos \theta$ | -1 | $-\frac{\sqrt{2}}{2}$ | 0 | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | -1 | 0 | 1 |
| $\tan \theta$ | 0 | 1 | | -1 | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | | $-\sqrt{3}$ | -1 | $-\frac{\sqrt{3}}{3}$ | 0 | | 0 |

2 The Derivative of Inverse Trigonometric Functions

2.1 $(f(x)^{-1})' =$ _____ .

2.2 Find the derivative of $y = \sin^{-1} x$.

2.3 Find the derivative of $y = \tan^{-1} x$.

2.4 Find the derivative of $y = \sec^{-1} x$.

2.5 $u = u(x)$

(a) $\frac{d}{dx}(\sin^{-1} u) = \underline{\hspace{2cm}}$, $|u| < 1$.

(b) $\frac{d}{dx}(\tan^{-1} u) = \underline{\hspace{2cm}}$.


(c) $\frac{d}{dx}(\sec^{-1} u) = \underline{\hspace{2cm}}$.

2.6 *Inverse Function-Inverse Cofunction Identities*

(a) $\cos^{-1} x = \underline{\hspace{2cm}} \Rightarrow \frac{d}{dx}(\cos^{-1} u) = \underline{\hspace{2cm}}$, $|u| < 1$


(b) $\cot^{-1} x = \underline{\hspace{2cm}} \Rightarrow \frac{d}{dx}(\cot^{-1} u) = \underline{\hspace{2cm}}$

(c) $\csc^{-1} x = \underline{\hspace{2cm}} \Rightarrow \frac{d}{dx}(\csc^{-1} u) = \underline{\hspace{2cm}}$, $|u| > 1$

 **Ex. 1** (example4, p408)

$$\frac{d}{dx}(\sin^{-1} x^2) =$$

sol:

 **Ex. 2** (example5, p410)

$$\frac{d}{dx}(\sec^{-1} 5x^4) =$$


sol:

3 Integration Formulas

$$3.1 \int \frac{du}{\sqrt{a^2 - u^2}} = \underline{\hspace{2cm}}, \quad u^2 < a^2$$

$$3.2 \int \frac{du}{a^2 + u^2} = \underline{\hspace{2cm}}$$


$$3.3 \int \frac{du}{u\sqrt{u^2 - a^2}} = \underline{\hspace{2cm}}, \quad |u| > a > 0$$

 **Ex. 3** (example6, p412)

$$(a) \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} =$$


$$(b) \int \frac{dx}{\sqrt{3-4x^2}} =$$

$$(c) \int \frac{dx}{\sqrt{e^{2x}-6}} =$$

 Ex. 4 (example7(a), p412)

Evaluate $\int \frac{dx}{\sqrt{4x - x^2}}$

sol:

 Ex. 5 (example7(b), p412)

Evaluate $\int \frac{dx}{4x^2 + 4x + 2}$

sol:

實習課練習 (EXERCISE 7.6)

□ Find the values.

9. $\sin(\cos^{-1}(\frac{\sqrt{2}}{2}))$

12. $\cot(\sin^{-1}(-\frac{\sqrt{3}}{2}))$

□ Find the derivative of y with respect to the appropriate variable.

22. $y = \cos^{-1}(1/x)$ $y = \sin^{-1}(3/t^2)$

33. $y = \ln(\tan^{-1} x)$

41. $y = x \sin^{-1} x + \sqrt{1 - x^2}$

42. $y = \ln(x^2 + 4) - x \tan^{-1}(\frac{x}{2})$

□ Evaluate the integrals.

44. $\int \frac{dx}{\sqrt{1 - 4x^2}}$

46. $\int \frac{dx}{9 + 3x^2}$

47. $\int \frac{dx}{x\sqrt{25x^2 - 2}}$

63. $\int_0^{\ln \sqrt{3}} \frac{e^x dx}{1 + e^{2x}}$

68. $\int \frac{dx}{\sqrt{2x - x^2}}$

72. $\int \frac{dy}{y^2 + 6y + 10}$

79. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$

83. $\int \frac{(\sin^{-1} x)^2 dx}{\sqrt{1 - x^2}}$

86. $\int \frac{dy}{(\sin^{-1} y)\sqrt{1 - y^2}}$

105-1 小考 (1) 題目

2016/10/03, Calculus Quiz (1), §2.1 ~ §2.5 (交回題目卷及答案卷)

- (a) What is the instantaneous speed at time t_0 ? (b) What is the Sandwich Theorem for ? (c) What is the precise definition of a limit? (d) What is the continuity of a function at a point?
- Find the slope of the curve $y = 2 - x^3$ at the given point $P(1, 1)$, and an equation of the tangent line at $P(1, 1)$. (Hint: start with the slope of a secant to the graph $y = f(x)$)
- If $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 1$, find (a) $\lim_{x \rightarrow 0} f(x)$, (b) $\lim_{x \rightarrow 0} \frac{f(x)}{x}$. (Hint: 需有算式)
- Use the precise definition of a limit to show that $\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$, where c is a constant.
- Find the limit: (a) $\lim_{t \rightarrow -3^+} \frac{\lfloor t \rfloor}{t}$, (b) $\lim_{t \rightarrow -3^-} \frac{\lfloor t \rfloor}{t}$. (c) $\lim_{t \rightarrow -4^+} (t - \lfloor t \rfloor)$, (d) $\lim_{t \rightarrow -4^-} (t - \lfloor t \rfloor)$
(Hint: $\lfloor x \rfloor$ is the greatest integer of x)(需有算式)
- For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0, \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x ?

105-1 小考 (2) 題目

2016/10/24, Calculus Quiz (2), §2.5 ~ §3.2 (交回題目卷及答案卷)(每題 20 分)

1. (a) What is the "Intermediate Value Theorem for Continuous Functions"? (b) Use the Intermediate Value Theorem to prove that the equation $x(x-1)^2 = 1$ has a solution.
2. Find the limit $\lim \left(\frac{1}{x^{1/3}} - \frac{1}{(x-1)^{4/3}} \right)$ as (a) $x \rightarrow 0^+$. (b) $x \rightarrow 0^-$. (c) $x \rightarrow 1^+$. (d) $x \rightarrow 1^-$.
3. Find the limit (a) $\lim_{x \rightarrow -\infty} (2x + \sqrt{4x^2 + 3x - 2})$. (b) $\lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$
4. (a) Find the horizontal and vertical asymptotes of the graph of $f(x) = \frac{1-x^2}{x^2+1}$. (b) Find the oblique asymptotes: $f(x) = \frac{x^2-4}{x-1}$.
5. (a) Using the definition to find the derivative of $f(x) = \sqrt{2x+1}$. (b) Find the tangent line to the above curve at $x = 1/2$.

105-1 小考 (3) 題目

2016/12/05, Calculus Quiz (3), §4.1 ~ §4.6 (交回題目卷及答案卷)(每題 20 分)

1. (a) What is the Mean Value Theorem? (b) What is the definition of a critical point of a function f ? (c) What is the definition of an inflection point of a function f ?
2. Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.
3. Two sides of a triangle have lengths a and b , and the angle between them is θ . What value of θ will maximize the triangle's area?
4. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2 . (ps. x_2 列式即可，不需算出答案)

105-1 小考 (4) 題目

2016/12/26, Calculus Quiz (4), §4.7 ~ §5.6 (交回題目卷及答案卷)(每題 25 分)

1. (a) What is the definition of an "antiderivative"? (b) What is a Riemann for f on the interval $[a, b]$? (c) What are The Fundamental Theorem of Calculus, Part I and Part II?
2. (a) Compute $\int_0^b \pi x^2 dx$ and (b) use the limit of Riemann sums to find the area under $y = \pi x^2$ over the interval $[0, b]$, $b > 0$.
3. (a) $\int_2^{-2} |x + 1| dx$, (b) Find dy/dx where $y = x \int_2^{x^2} \sin(t^3) dt$.
4. (a) $\int \frac{1}{\sqrt{2x}(1 + \sqrt{2x})^2} dx$, (b) Find the area of the region between the curve $y = 5 - x^2$ and the line $y = 1$.

國立臺北大學 105 學年度第 1 學期 期中 考試命題紙

考試科目：微積分

開課班別：通訊 1/電機 1/智財學程

命題教授：吳漢銘

考試日期：11 月 7 日（一）10:00~11:30

※准帶項目打「O」，否則打「X」

1. 需加發計算紙或答案紙請在試題內封袋備註。
2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

本試題共 1 頁，印刷份數：82 份

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備註：注意事項要看!!

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注意事項：(1) 答案卷請寫上姓名及學號。(2) 請按題號順序書寫。(3) 每一題號需置於答案卷最左邊。(4) 可用鉛筆。(5) 需要計算過程。(6) 同時交回答案卷、題目卷、計算紙。

1. (a) What is the instantaneous rate of change of $f(x)$ at $x = c$. (b) What is the derivative of the function $f(x)$ with respect to x . (c) What does it mean that a function $f(x)$ is differentiable at $x = c$?

2. Let $f(x)$ and $g(x)$ be differentiable functions. Use the definition of the derivative of a function to prove The Quotient Rule.

3. Let $s(t) = \frac{t-1}{t+1}$. Use calculus (the definition of the derivative) to find the instantaneous rate of change of $s(t)$ at $t = -1/2$.

4. Find the derivative of f . (a) $f(x) = \sqrt[3]{(3x^2 - 4)^5 + 3x}$. (b) $f(x) = \frac{3x-2}{(2x-1)^2}$

5. Find $h'(0)$ if $h(x) = \left[\frac{g(x) - x}{3 + g(x)} \right]^2$, where $g(0) = 3$ and $g'(0) = -2$.

6. (a) What is the definition of the continuity of a function f at the point c ? (b) What is the definition of the derivative of a function f at x ? (c) What does "a function is differentiable at x " mean?

7. (a) Find $\lim_{x \rightarrow 2} f(x)$, where $f(x) = \begin{cases} \frac{x}{3} - 2, & x \leq 2 \\ -2x + 5, & x > 2 \end{cases}$ (b) Find $\lim_{x \rightarrow -2} \frac{[2x] + 2}{2x + 2}$, where $[\cdot]$ is the Gauss' symbol.

8. Find dy/dx for the equation $xy^3 + (2y)^2 - 5y = \frac{(x^2 - 9)}{\sqrt{x^2 + 2}}$.

9. Find the asymptotes of the graph of $f(x) = \frac{1 - x^2}{8 - (\sqrt{2}x)^2}$.

注意：1、考試求公平及公正，請同學務必自律，維護學校與學生之榮譽。

2、考試時不得有交談、窺視、夾帶、抄襲、傳遞、代考或其它作弊等舞弊行為，考畢務必交卷，不得攜卷出場，違者依考場規則議處。

國立臺北大學 105 學年度第 1 學期 期末 考試命題紙

考試科目：微積分

開課班別：通訊 1/電機 1/智財學程

命題教授：吳漢銘

考試日期：1 月 9 日（一）10:00~11:30

※准帶項目打「O」，否則打「X」

1. 需加發計算紙或答案紙請在試題內封袋備註。
2. 為環保節能減碳，試題一律採雙面印刷，如有特殊印製需求，請註記：

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備註：注意事項要看!!

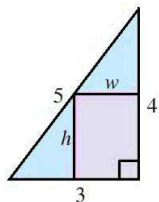
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注意事項：(1) 答案卷請寫上姓名及學號。(2) 請按題號順序書寫。(3) 每一題號需置於答案卷最左邊。(4) 可用鉛筆。(5) 需要計算過程。(6) 同時交回答案卷、題目卷、計算紙。

1. (10 分) (a) What is a Riemann sum for a function f ? (b) What are "The Fundamental Theorem of Calculus" Part 1 and Part 2?

2. (15 分) Sketch the graph of $f(x) = \frac{x^2 - 3}{x - 2}$. (Find where the curve is increasing and where it is decreasing. Find the local extreme points. Find the points of inflection, if any occur, and determine the concavity of the curve. Identify any asymptotes.)

3. (10 分) Determine the dimensions of the rectangle of largest area that can be inscribed in the right triangle shown in the following figure.



4. (15 分) (a) Compute $\int_a^b x^2 dx$. (b) Use the limit of Riemann sums to find the area under $y = x^2$ over the interval $[a, b]$, $b > a$.

5. (10 分) Find dy/dx , if $y = \int_2^{7x^2} \sqrt{2 + \cos^3 t} dt$.

6. (10 分) Find the areas of the regions enclosed by the lines and curves. $y = |x^2 - 4|$, and $y = (x^2/2) + 4$.

7. (10 分) $\int x^5 \sqrt{2x^3 + 1} dx$.

8. (20 分) By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about (a) the x -axis. (b) the y -axis.

9. 加分題 (20 分): (a) 微積分助教姓名? (b) 助教研究室在哪一間? (c) 微積分課本書名為何? (d) 微積分課本作者及版本為何?

注意：1、考試求公平及公正，請同學務必自律，維護學校與學生之榮譽。

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