

## THOMAS' CALCULUS (12/E)

## 8.7 Improper Integrals

開課班級: 土木系工設組1A1B 微積分

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# 1 Infinite Limits of Integration

## 1.1 Definitions: Type I Improper Integrals

Integrals with \_\_\_\_\_ of integration are **improper integrals of Type I**.

(a) If  $f(x)$  is continuous on  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \underline{\hspace{2cm}}$$

(b) If  $f(x)$  is continuous on  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \underline{\hspace{2cm}}$$

(c) If  $f(x)$  is continuous on  $(-\infty, \infty)$  then

$$\int_{-\infty}^{\infty} f(x) dx = \underline{\hspace{2cm}}$$


where  $c$  is any real number. In each case, if the limit is \_\_\_\_\_ we say that the improper integral \_\_\_\_\_ and that the \_\_\_\_\_ is the **value** of the improper integral. If the limit fails to exist, the improper integral \_\_\_\_\_ (the area under the curve is infinite).

## 1.2 Examples:

(a) Upper limit:  $\int_1^{\infty} \frac{\ln x}{x^2} dx = \underline{\hspace{2cm}}$


(b) Lower limit:  $\int_{-\infty}^0 \frac{dx}{1+x^2} = \underline{\hspace{2cm}}$

(c) Both limits:  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \underline{\hspace{2cm}}$

 **Ex. 1** ..... (example1, p479)


Is the area under the curve  $y = (\ln x)/x^2$  from  $x = 1$  to  $x = \infty$  finite? If so, what is it?

*sol:*

 **Ex. 2** ..... (example2, p479)

Evaluate  $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ .

*sol:*

 **Ex. 3** ..... (example2, p480)

For what values of  $p$  does the integral  $\int_1^{\infty} \frac{1}{x^p} dx$  converge? When the integral does converge, what is its value?

*sol:*

## 2 Integrands with Vertical Asymptotes

### 2.1 Definitions: Type II Improper Integrals

Integrals of functions that become \_\_\_\_\_ within the interval of integration are **improper integrals of Type II**.

(a) If  $f(x)$  is continuous on  $(a, b]$  and is discontinuous at  $a$  then

$$\int_a^b f(x) dx = \underline{\hspace{2cm}}$$

(b) If  $f(x)$  is continuous on  $[a, b)$  and is discontinuous at  $b$ , then

$$\int_a^b f(x) dx = \underline{\hspace{2cm}}$$

(c) If  $f(x)$  is discontinuous at  $c$ , where  $a < c < b$ , and continuous on  $[a, c) \cup (c, b]$  then

$$\int_a^b f(x) dx = \underline{\hspace{2cm}}$$


In each case, if the limit is finite we say the improper integral \_\_\_\_\_ and that the limit is the value of the improper integral. If the limit does not exist, the integral \_\_\_\_\_.

### 2.2 Examples:

(a) Upper endpoint:  $\int_0^1 \frac{dx}{(x-1)^{2/3}} = \underline{\hspace{2cm}}$


(b) Lower endpoint:  $\int_1^3 \frac{dx}{(x-1)^{2/3}} = \underline{\hspace{2cm}}$

(c) Interior point:  $\int_0^3 \frac{dx}{(x-1)^{2/3}} = \underline{\hspace{2cm}}$

 **Ex. 4** ..... (example4, p482)

Investigate the convergence of  $\int_0^1 \frac{1}{1-x} dx$ .

*sol:*

 **Ex. 5** ..... (example5, p482)

Evaluate  $\int_0^3 \frac{dx}{(x-1)^{2/3}}$ .

*sol:*

### 3 Tests for Convergence and Divergence

#### 3.1 Theorem 1: Direct Comparison Test

Let  $f$  and  $g$  be continuous on  $[a, \infty)$  with \_\_\_\_\_ for all  $x \geq a$ .


Then

(a)  $\int_a^\infty f(x) dx$  \_\_\_\_\_ if  $\int_a^\infty g(x) dx$  \_\_\_\_\_.

(b)  $\int_a^\infty g(x) dx$  \_\_\_\_\_ if  $\int_a^\infty g(x) dx$  \_\_\_\_\_.


#### 3.2 Theorem 2: Limit Comparison Test

If the positive functions  $f$  and  $g$  are continuous on  $[a, \infty)$  and if \_\_\_\_\_,  $0 < L < \infty$ , then \_\_\_\_\_ and \_\_\_\_\_ both \_\_\_\_\_ or both \_\_\_\_\_.

 **Ex. 6** ..... (example6, p483)


Does the integral  $\int_1^\infty e^{-x^2} dx$  converges?

*sol:*

 **Ex. 7** ..... (example7, p484)


(a)  $\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$  converges because

(b)  $\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}} dx$  diverges because

 **Ex. 8** ..... (example8, p485)

Show that  $\int_1^{\infty} \frac{dx}{1+x^2}$  converges and find the integral value.

*sol:*

 **Ex. 9** ..... (example9, p485)

Investigate the convergence of  $\int_1^{\infty} \frac{1 - e^{-x}}{x} dx$ .

*sol:*

## 實習課練習 (EXERCISE 8.7)

□ Evaluate the integrals.

2.  $\int_1^{\infty} \frac{dx}{x^{1.001}}$

7.  $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

10.  $\int_{-\infty}^2 \frac{2 dx}{x^2 + 4}$

$$13. \int_{-\infty}^{\infty} \frac{2x \, dx}{(x^2 + 1)^2}$$

$$17. \int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$$

$$22. \int_0^{\infty} 2e^{-\theta} \sin \theta \, d\theta$$

$$25. \int_0^1 x \ln x \, dx$$

$$30. \int_2^4 \frac{dt}{t\sqrt{t^2 - 4}}$$

$$32. \int_0^2 \frac{dx}{\sqrt{|x - 1|}}$$

□ Test the integrals for convergence.

$$39. \int_0^{\ln x} x^{-2} e^{-1/x} dx$$

$$45. \int_{-1}^1 \ln |x| dx$$

$$54. \int_2^{\infty} \frac{x dx}{\sqrt{x^4 - 1}}$$

$$58. \int_2^{\infty} \frac{1}{\ln x} dx$$

$$61. \int_1^{\infty} \frac{1}{\sqrt{e^x - x}}$$