計算錯誤扣2分 忘了寫正負號、lim、ln、+C扣1分
定積分寫成不定積分扣4分
亂寫的地方不給分 設變數沒有代回扣2分

求f(x) 5分，求limf(x)5分

1. $f(x)=\left(\frac{x^2+1}{x+2}\right)^{1/x}$, $\ln f(x)=\ln\left(\frac{x^2+1}{x+2}\right)^{1/x}=\frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)$

$$\lim_{x\to\infty}\ln f(x)=\lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{x^2+1}{x+2}\right)=\lim_{x\to\infty}\frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x}=\lim_{x\to\infty}\frac{\ln(x^2+1)-\ln(x+2)}{x}$$

$$=\lim_{x\to\infty}\frac{\frac{2x}{x^2+1}-\frac{1}{x+2}}{1}=\lim_{x\to\infty}\frac{x^2+4x-1}{(x^2+1)(x+2)}=\lim_{x\to\infty}\frac{x^2+4x-1}{x^3+2x^2+x+2}=\lim_{x\to\infty}\frac{2x+4}{3x^2+4x+1}=\lim_{x\to\infty}\frac{2}{6x+4}=0$$

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x+2}\right)^{1/x}=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^0=1$$

2. $u=\sin^{-1}x$, $du=\frac{dx}{\sqrt{1-x^2}}$ (3分)

$$\int\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx=\int e^u du=e^u+C=e^{\sin^{-1}x}+C$$ (7分)

3. $u=\sin^{-1}(x^2)$, $du=\frac{2x\,dx}{\sqrt{1-x^4}}$, $dv=2x\,dx$, $v=x^2$ (3分)

$$\int_0^{\frac{1}{\sqrt{2}}}2x\sin^{-1}(x^2)\,dx=\left[x^2\sin^{-1}(x^2)\right]_0^{\frac{1}{\sqrt{2}}}-\int_0^{1/\sqrt{2}}x^2\frac{2x\,dx}{\sqrt{1-x^4}}=\left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right)+\int_0^{1/\sqrt{2}}\frac{d(1-x^4)}{2\sqrt{1-x^4}}$$ (4分)

$$=\frac{\pi}{12}+\left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}}=\frac{\pi}{12}+\sqrt{\frac{3}{4}}-1=\frac{\pi+6\sqrt{3}-12}{12}$$ (3分)

4. $\int\eta\cos^{\eta}t\,dt=\eta\int\cos t\cos^{6}t\,dt=\eta\int\cos t(1-3\sin^2t+3\sin^4t-\sin^6t)\,dt$ (3分) (4分)

$$=\eta\left[\int\cos t\,dt-3\int\sin^2t\cos t\,dt+3\int\sin^4t\cos t\,dt-\int\sin^6t\cos t\,dt\right]$$

$$=\eta\left(\sin t-3\frac{\sin^3t}{3}+3\frac{\sin^5t}{5}-\frac{\sin^{\eta}t}{\eta}\right)+C$$

$$=\eta\sin t-\eta\sin^3t+\frac{21}{5}\sin^5t-\sin^{\eta}t+C$$ (3分)

5. $\int_1^e\frac{dy}{y\sqrt{1+(\ln y)^2}}=\int_0^{\frac{\pi}{4}}\frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}d\theta=\int_0^{\frac{\pi}{4}}\sec\theta\,d\theta=\left[\ln\left|\sec\theta+\tan\theta\right|\right]_0^{\frac{\pi}{4}}=\ln(1+\sqrt{2})$ (4分) (3分)

$y=e^{\tan\theta}$, $0\le\theta\le\frac{\pi}{4}$, $dy=e^{\tan\theta}\sec^2\theta\,d\theta$, $\sqrt{1+(\ln y)^2}=\sqrt{1+\tan^2\theta}=\sec\theta$ (3分)

Express the integrand as a sum of partial fractions(5分)

6. $\int \frac{x^4+81}{x(x^2+9)^2}dx$ ; $\frac{x^4+81}{x(x^2+9)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+9} + \frac{Dx+E}{(x^2+9)^2}$

$x^4+81 = A(x^2+9)^2 + (Bx+C)x(x^2+9) + (Dx+E)x$

$= A(x^4+18x^2+81) + (Bx^4+Cx^3+9Bx^2+9Cx) + Dx^2+Ex$

$= (A+B)x^4 + Cx^3 + (18A+9B+D)x^2 + (9C+E)x + 81A$

$\Rightarrow A=1, B=0, C=0, D=-18, E=0$

Evaluate the integral(5分)

$\int \frac{x^4+81}{x(x^2+9)^2}dx = \int \frac{dx}{x} - 18\int \frac{x\,dx}{(x^2+9)^2} = \ln|x| + \frac{9}{x^2+9} + C$

7. $\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \int_0^1 \frac{dx}{\sqrt{1-x}} + \int_1^2 \frac{dx}{\sqrt{x-1}} = \lim_{b\to 1^-}[-2\sqrt{1-x}]_0^b + \lim_{c\to 1^+}[2\sqrt{x-1}]_c^2$ (4分) (3分)

$= \lim_{b\to 1^-}(-2\sqrt{1-b}) - (-2\sqrt{1-0}) + 2\sqrt{2-1} - \lim_{c\to 1^+}(2\sqrt{c-1}) = 0+2+2-0 = 4$ (3分)

Find a formula for the nth partial sum of the series(5分)

8. $S_n = (\ln\sqrt{2} - \ln\sqrt{1}) + (\ln\sqrt{3} - \ln\sqrt{2}) + \cdots + (\ln\sqrt{n} - \ln\sqrt{n-1}) + (\ln\sqrt{n+1} - \ln\sqrt{n})$

$= \ln\sqrt{n+1} - \ln\sqrt{1} = \ln\sqrt{n+1}$

解釋(2分) converges or diverges(3分)

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln\sqrt{n+1} = \infty$ , diverges

9. $\sum_{n=1}^{\infty}(1-\frac{1}{n})^n$ ; 解釋(7分) $\lim_{n\to\infty}(1-\frac{1}{n})^n = \lim_{n\to\infty}(1+\frac{-1}{n})^n = e^{-1} \neq 0$ , diverges converges or diverges(3分)

需寫出p>1及<=1的歛散並解釋，少一部分扣3分 沒解釋清楚扣1分

10. $u=\ln x, du=\frac{dx}{x} \to \int_2^{\infty}\frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} u^{-p}du = \lim_{b\to\infty}[\frac{u^{-p+1}}{-p+1}]_{\ln 2}^b = \lim_{b\to\infty}(\frac{1}{1-p})[b^{-p+1} - (\ln 2)^{-p+1}]$

if $p=1$, $\int_2^{\infty}\frac{dx}{x(\ln x)} = \lim_{b\to\infty}[\ln(\ln x)]_2^b = \lim_{b\to\infty}[\ln(\ln b) - \ln(\ln 2)] = \infty$

$\to \begin{cases} \frac{1}{p-1}(\ln 2)^{-p+1}, & p>1 \\ \infty, & p\leq 1 \end{cases}$

需解釋積分為收歛(7分)

11. $e^{-x^2} \leq e^{-x}$ for $x \geq 1$, $\int_1^{\infty} e^{-x}dx = \lim_{b\to\infty}[-e^{-x}]_1^b = \lim_{b\to\infty}(-e^{-b}+e^{-1}) = e^{-1}$

$\int_1^{\infty} e^{-x^2}dx$ converge (by comparison test for improper integrals)

$\sum_{n=0}^{\infty} e^{-n^2} = \sum_{n=1}^{\infty} e^{-n^2} + 1$ converge (by intergral test)

利用積分收歛證明級數和收歛(3分)