Bayesian Resolution of the "Exchange Paradox "

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1. THE PARADOX

- A swami puts m dollars in one envelope, and $2m$ dollars in another.
- He hands one envelope to you and one to your opponent, so that the probability is 0.5 that you get either envelope.
- You open your envelope and find x dollars. Let Y be the amount in your opponent's envelope.
- With a gleam in your eye, you offer to trade envelopes with your opponent.
- Since she has made the same calculation, she readily agrees.

1. THE PARADOX

- x: the amount in your envelope
- Y : the amount in your opponent's envelope

$$Y = \frac{x}{2} \text{ or } 2x$$

$$E(Y) = \frac{1}{2} \cdot \frac{x}{2} + \frac{1}{2} \cdot 2x = \frac{5}{4}x$$

1. THE PARADOX

- One should always trade is intuitively unreasonable while the method of arriving at the rule seems very reasonable.
- Nani?

2. A BAYESIAN RESOLUTION

- A Bayesian approach uses prior information to model beliefs about the value of m.
- Some prior information for the value of m certainly exists.

2. A BAYESIAN RESOLUTION

- M: subjectively random amount of money placed in the first envelope
- $g(m)$: the prior density for M
- X: the random amount of money in your envelope
- Since X equals either M or $2M$,
- on observing $X = x$, M can take on only two values, x and $x/2$

2. A BAYESIAN RESOLUTION

$$P(M = x|X = x) = \frac{P(X = x|M = x)g(x)}{P(X = x|M = x)g(x) + P\left(X = x\middle|M = \frac{x}{2}\right)g\left(\frac{x}{2}\right)}$$

$$= \frac{g(x)}{g(x) + g\left(\frac{x}{2}\right)}$$

$$P\left(M = \frac{x}{2}\middle|X = x\right) = \frac{g\left(\frac{x}{2}\right)}{g(x) + g\left(\frac{x}{2}\right)}$$

2. A BAYESIAN RESOLUTION

$$E(W|Trade) = E(Y|X = x) = \frac{g\left(\frac{x}{2}\right)}{g(x) + g\left(\frac{x}{2}\right)} \cdot \frac{x}{2} + \frac{g(x)}{g(x) + g\left(\frac{x}{2}\right)} \cdot 2x$$

- When $g\left(\frac{x}{2}\right) = 2 \cdot g(x)$, $E(W|Trade) = x$
- Therefore, if $g\left(\frac{x}{2}\right) > 2 \cdot g(x)$, it is optimal to keep the envelope
- if $g\left(\frac{x}{2}\right) < 2 \cdot g(x)$, it is optimal to trade envelopes.

3. Reference

- Christensen, Ronald, and Jessica Utts. "Bayesian Resolution of the 'Exchange Paradox.'" *The American Statistician*, vol. 46, no. 4, 1992, pp. 274–76. *JSTOR*, https://doi.org/10.2307/2685310 . Accessed 7 Jan. 2024.

Thank youuuu